OVERHEADs Cointegration Lecture, 3/29/99

If y_t has a stationary invertible ARMA representation after differencing d times, it is said to be integrated of order d, denoted by $y_t \sim I(d)$ if $(1 - L)^d y_t$ is stationary or I(0).
Note the root of the polynomial $(1 - L)^d$ in L is 1, unit root! There are d roots each =1.
If d=0 we have I(0) $\Rightarrow$ levels are stationary, e.g. white noise from $N(0, \sigma^2)$.
Random walk processes exhibit long memory innovations are permanent and $\psi_k \to 1$ for all k as $t \to \infty$. Vaiance of I(1) is infinite.

If $y_t \sim I(1)$ and if $x_t \sim I(0)$, then the linear combination $z_t = x_t + y_t \sim I(1)$, (4)
If $y_t \sim I(d)$ and a and b are constants, then the linear transformation $(a\, y_t + b) \sim I(d)$, (5)
To verify (4) note that the infinite variance of I(1) series will eventually dominate. We would also expect that:

If $y_t \sim I(1)$ and if $x_t \sim I(1)$, then the linear combination $z_t = a\, x_t + b\, y_t \sim I(1)$ (6)
If $y_t \sim I(d)$ and if $x_t \sim I(d)$, then the linear combination $z_t = a\, x_t + b\, y_t \sim I(d)$ (7)

Co-integration

short and long term interest rates, household incomes and expenditures, commodity prices (gold) in geographically separated markets, capital appropriations and expenditures by business. Adam Smith showed that the invisible hand of the market forces will tend to force the excess demand, $z_t = (q_d - q_s) \to 0$ in the dynamic sense implied by $z_t \sim I(0)$ or $N(0, \sigma^2)$, then mean is zero! hover around zero stochastically!
The joint trending of the two variables may mean that the long-run components cancel each other in some sense. q_d is I(1), q_s is I(1) but $(q_d - q_s)$ is I(0) so cointegrated!

DEFINITION of Co-integration: The components of a $k \times 1$ vector y_t with $k \geq 2$ time series are said to be co-integrated of order (d, b) denoted by $y_t \sim CI(d,b)$ if $y_t \sim I(d)$ and there exist $r \geq 1$ "co-integrating vectors" α_i ($\neq 0$) of dimension $k \times 1$ defining r linear combinations which are integrated of order $d - b$, or

$$z_{it} = \alpha_i' y_{it} \sim I(d - b), \quad b>0,\ i = 1,2, \ldots, r, \quad \text{where } d>b>0 \qquad (1)$$

where the elements of y_t are denoted by y_{jt} with $j=1,2, \ldots, k$.
Multivariate ARMA models for y_t can be used to write the following matrix equation.

$$\Delta y_t = G(L)\, a_t \qquad (2)$$

where a_t denotes a white noise process. Engle and Granger(1987) show that when the elements of y_t are co-integrated, the matrix G(1) obtained by simply replacing the operator L by unity does not have a full rank. Hence the following <u>multivariate</u> (Wold) decomposition is plausible:

$$y_t = \sum_{j=0}^{\infty} G_j\, a_{t-j}, \qquad G_0 = I \qquad (3)$$

where G_j are called Green's function matrices of dimension $k \times k$ each. It is shown in the literature that the representation (3) is unique under certain regularity conditions. The novel feature in the present context of co-integration is that we have Δy_t instead of y_t on the left side of (3).
vector autoregressive (VAR) models

$$\Phi(L)\, y_t = (I_k - \Sigma_{j=1}^{p} \Phi_j L^j)\, y_t = a_t \qquad (8)$$

Observe that $\Phi(L)$ is also a $k \times k$ matrix whose elements are polynomials in the lag operator L. If we try a VAR representation of the differenced data we have

$$\Phi^*(L)\,(1 - L)\, y_t = a_t \qquad (9)$$

where the star notation $F^*(L)$ œ $(I_k • D_{j=1}^{p-1} F_j L^j)$ and $F_j^* = • D_{m=j+1}^{p} F_m$

Gonzalo (1994, J of Etr. Vol. 60, p. 203) compares five methods of estimating cointegration: OLS, NLS, Maximum Likelihood in an Error Correction model (MLECM), Principal Components (PC) and Canonical Correlation (CC). He finds that Johansen's MLECM is the best.

Error correction model (ECM) relates $?y_t$, the change in one variable is a function of past equilibrium errors (u_t) and past changes in both variables:

$?x_t = \#_1(y_{t-1} • ! x_{t-1})$ + p– lags of $?x_t$ and $?y_t$ + white noise errors

$?y_t = \#_2(y_{t-1} • ! x_{t-1})$ + p– lags of $?x_t$ and $?y_t$ + white noise errors

where the errors may be correlated. The terms $(y_{t-1} • ! x_{t-1})$ are called error correction terms, which are I(0) if the variables are cointegrated.

Engle and Granger (1987, Etrica p254) if at least one unit root

$$A(L)\,(1–L)\,y_t = • \# z_{t-1} + u_t,$$

where u_t is stationary multivariate disturbance, and there are exactly r cointegrating relations (equilibria) satisfying $z_r = ! y_r$. This is a multivariate representation without assuming a subset of variables to be (weakly) exogenous. Also, ! need not be a set of constants arising from economic theory. It is treated as a parameter. Harvey (1990, p. 290) presents another way of thinking about this is in terms of a special case:

$$?y_t = a_0 + a_1\,?y_{t-1} + b_0\,?x_t + b_1\,?x_{t-1} + (! • 1)\{\,y_{t-1} • \frac{"}{1 • !}\,x_{t-1}\} + \%_t$$

If ! is close to 1, one may concentrate on the short-run and ignore the long-run term {.}. When !<1 the *level* terms play the crucial role of *error correction* in the following manner. If a shock causes y_{t-1} to become too large to ensure equilibrium, the {.} term becomes positive, but the presence of (!–1) which is negative for !<1 will bring down the left side $?y_t$ and slow down the short-run growth in y.