

To Econometrics II students:

I have recently published the following papers on the unit root stuff we discussed in the class. It may be of interest to some of you.

“Closed Forms for Asymptotic Bias and Variance in Autoregressive Models with Unit Roots,” joint with L. R. Shenton, *Journal of Computational and Applied Mathematics* 61(1995) 231-243. This is a highly mathematical (pure) journal. This deals specifically with the unit root case. The exact bias has a closed form:

$$E(\hat{\alpha})|_{\alpha=1} = 1 + \frac{2}{N} - \frac{(N+1)(N-5)}{8N} \int_0^{\infty} \frac{e^{-t} \Phi(t, N) dt}{\sqrt{(1+e^{-4t})}}, \quad (N > 5) \quad (2.13)$$

where $\Phi(t, N)$ appearing in the integrand is written as a function of $m = \frac{4t}{N}$

$$\Phi(m) = \nu_1 m + \nu_3 m^3 + \nu_5 m^5 + \dots$$

an odd series in descending powers of N . The series for $\Phi(m)$ may be set up using the symbolic manipulation software (e.g. Mathematica or Derive).

$$\begin{aligned} \Phi(m) = & 2\sqrt{2} [m + 0.2291666667 m^3 + 0.01809895833 m^5 + 0.0006622798859 m^7 + \\ & 0.0001493652451 m^9 + 1.787248666 (10)^{-7} m^{11} + 3.966156353 (10)^{-9} m^{13} - 1.217705059 \\ & (10)^{-10} m^{15} + 1.034724986 (10)^{-11} m^{17} - 7.904984013 (10)^{-13} m^{19} + 6.253205271 (10)^{-14} \\ & m^{21} - 5.050103874 (10)^{-15} m^{23} + 4.153101875 (10)^{-16} m^{25} - 3.468950664 (10)^{-17} \\ & m^{27} + 2.936805131 (10)^{-18} m^{29} - 2.515727180 (10)^{-19} m^{31} + 2.177462861 (10)^{-20} m^{33} \\ & - 1.902063858 (10)^{-21} m^{35} + 1.675186876 (10)^{-22} m^{37} - 1.486195111 (10)^{-23} m^{39} \\ & + 1.327311473 (10)^{-24} m^{41} - 1.192549973 (10)^{-25} m^{43} + 1.077340284 (10)^{-26} m^{45} \\ & - 9.781320220 (10)^{-28} m^{47} + 8.821359341 (10)^{-29} m^{49} - 8.171381514 (10)^{-30} m^{51} + O(m^{53}). \end{aligned}$$

I am indebted to Dr. Byers of C.D.C. for these computations. The curious thing for mathematicians is that first seven terms in the expression in powers of $m = \frac{4t}{N}$ are positive and then the pattern switches signs. This phenomenon is very rare, and it warns against standard practice of using only first few terms in expansions. Of course a practitioner may say who cares when m (quantity proportional to the reciprocal of the sample size is raised to the power 15 or more.) IT also confirms the mathematical curiosity and interest associated with the unit root case.

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Exact Moments for Autoregressive and Random Walk Models
for a Zero or Stationary Initial Value.

H. D. Vinod and L. R. Shenton

Fordham University and University of Georgia

Econometric Theory, Vol 12, 1996, pages 481-499.

Summary:

For a first-order autoregressive AR(1) model with zero initial value, $x_t = \alpha x_{t-1} + \epsilon_t$, we provide the bias, mean squared error (MSE), skewness and kurtosis of the maximum likelihood (ML) estimator $\hat{\alpha}$. Brownian motion approximations by Phillips (1977 *Etrica* p 463; 1978 *Biometrika* p 91, 1987 *Etrica*, p 277), Phillips and Perron (1988, *Biom* p 335), Perron (1991 *Etric Theory* p 236, 1991b *Etrica* p 211), among others, yield an elegant unified theory, but do not yield convenient formulas for calibration of skewness and kurtosis. In addition to the usual stationary case $|\alpha| < 1$, we include the unstable $|\alpha| = 1$ case of the random walk model. For the $|\alpha| < 1$ case, we give new exact results for White's (1961) model B, where the initial value x_0 is a normal random variable $N(0, \sigma^2/(1-\alpha^2))$. Our expressions are exact for small samples computed by relatively reliable Gaussian quadrature methods, rather than approximate ones in powers of n^{-1} or α^2 .

The basic idea in this paper is to write the OLS/Maximum likelihood estimator. Write its likelihood function, then the moment generating function (mgf). White(1961, *Biom*, p.85) showed that the mgf involves a determinant of a tridiagonal matrix with very special form. The determinant satisfies a 2nd order Difference equation (not stochastic difference eq. just a diff. eq.) We use various mathematical tricks to solve this difference eq. analytically. Faa di Bruno's formula (1876, not a misprint, it is 18th century. The journal's copy editors twice changed our 1876 to 1976 thinking it was a typo. We had to call them. We all know that much of truly useful pure math was invented in 19th not 20th century) for nth derivative of the function of a function was used. The continuous time approximation (CTA) used by Phillips and Perron and others needs to be calibrated for higher moments. For the bias, n has to be larger than 100 for CTA to yield good approximation to our exact results. The calibration of higher moments is unknown and a possible topic for a dissertation.

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"A Note on Convergence when the Root is Greater than Unity"

H. D. Vinod and B. D. McCullough,

Journal of Quantitative Economics, 11, 1995, 251-256.

In the explosive case $|\alpha| > 1$ the density becomes Cauchy whose mean does not exist, so the bias cannot be computed. However we bound the point wise difference between estimate $\hat{\alpha}$ and their true value α and prove that the convergence is very fast.

We prove that $|\hat{\alpha} - \alpha|$ converges to $(1/\pi)[\alpha^2 - 1]/[|\alpha|^n]$
Our simulation shows that the bias and variance decline for the explosive case and the usual formula are quite unreliable.