8 Amortization: borrowing

The Present amortization formula

In this section, we want to look at the second kind of AMORTIZATION (1.6.1), loans, where we receive the lump sum *before* making a series of equal payments at regular intervals. Here our model example will be a home mortgage. Suppose that, in addition to my down payment, I need \$100,000 to purchase a house. The bank will lend me this money as a lump sum mortgage — say at a nominal rate of 7.5% a year — and in return I agree to make payments at the end of every month for a term of 30 years. The basic question here is, "What should my monthly payment be?". Closely related is the question I might ask myself before starting to look for a home: "If I can afford to budget about \$800 a month for my mortgage, how big a mortgage will the bank give me?".

Why did I separate the savings and loan amortizations? The answer is not that they are mathematically very different. In fact, we'll be able to apply the formula derived for savings accounts with very little change. What's different about loans is that, in most cases, they do not involve the uncertainty of savings accounts. When we take out a loan, we general know going in what the interest rate will be and hence exactly each payment must be to amortize the loan. This kind of predictability is rare for savings and investments, as we have seen. The result is that we can answer precisely questions about loans in which for an investment, we'd have to use a guess or estimate.

To begin with, we are going to figure out a formula for relating the balance B and payment D of a loan *with no calculation*! Before reading on, go back and look at the derivation of the FUTURE AMORTIZATION FORMULA (1.6.9). We had to work hard to get that formula even if it turned out to be very simple at the end. So you should be be impressed that we can figure out the analogous formula for loans with no work. I'm lying, of course: what we'll really do is shake the FUTURE AMORTIZATION FORMULA (1.6.9) until the PRESENT AMORTIZATION FORMULA (1.8.1) drops out making the work we have already done pay double dividends. But to a mathematician, that's always like getting something for nothing, the charm of the subject. This is also where introducing the sum and balance notation (S and B) will pay off.

So, how are the payment and the balance of a loan related? The easiest way to see the answer is to ask: "What's the difference between an investment and a loan if the interest rate, period, term and deposit of each are the same?". In other words, suppose we make equal payments or deposits of $\$D$ at equal periods — say m times a year — for a term of T periods into an account that earns a periodic interest rate p. So far, we could either be describing what we do to accumulating money in a savings or investment account or what we do to repay a loan. What's the difference? Not a whole lot. In both cases, what we want to calculate is a single lump sum of money which represents the combined value of all the deposits. The only difference is *the point in time* at which that lump sum exists. In a savings account, the lump sum is the *final sum* $S = S_T$ in the account at the *end* of the term of the amortization. In a loan repayment, the lump sum is the *initial balance* $B = B_0$ of the loan at the *start* of the term of the amortization. To repeat, both these lump sums represent the combined value of the series of T deposits of $\$D$. Lets call this A to be neutral. The only difference is the moment in time at which this combined value is computed. The initial balance B of a loan is the value of the amount A at the *start* of the term of the amortization, after 0 periods have passed: in other words, $B = A_0$. The final sum S in a a savings account is the value of the amount A at the *end* of the term of the amortization, after all T periods have passed: in other words, $S = A_T$.

But the COMPOUND INTEREST FORMULA (1.2.4) tells us how A_0 and A_T are related: $A_T = A_0 \cdot (1+p)^T$. Thus, we conclude that $S = B \cdot (1+p)^T$, or multiplying both sides by $(1+p)^{-T}$, that $B = S \cdot (1+p)^{-T}$. Bingo! Now all we have to do is plug in the value of S given by the FUTURE AMORTIZATION FORMULA (1.6.9), munge a few exponents and we find that

$$B = S \cdot (1+p)^{-T} = D\left(\frac{(1+p)^{T}-1}{p}\right) \cdot (1+p)^{-T} = D\left(\frac{1-(1+p)^{-T}}{p}\right)$$

PRESENT AMORTIZATION FORMULA (1.8.1) If a loan which earns compound interest at a periodic rate p is repaid by a series of payments made at the end of each of T periods, then the initial balance B of the loan and the amount D of each payment are related by

$$B = D\left(\frac{1-(1+p)^{-T}}{p}\right) \quad \text{and} \quad D = B\left(\frac{p}{1-(1+p)^{-T}}\right).$$

Before we use this, a warning is in order. The key observation that let us derive this formula with no calculation was that $B_0 = A_0$ and that $S_T = A_T$. What about all the intermediate balances B_i and sums S_i after i deposits or payments are made? Can we relate these to the value A_i of the amount A after i periods? No! The point is that A and hence any A_i involves *all* the payments and this is only true the *initial* balance B_0 and *final* sum S_T. But not to worry. In a moment, we'll see that the PRESENT AMORTIZATION FORMULA (1.8.1) actually can tell us about all the intermediate balances too.

Working with the present amortization formula

Once again, the formula is beautifully simple to use. Here are the answers to the mortgage questions at the start of this section.

Example (1.8.2) I am going to borrow \$100,000 from the bank at 7.5% interest and make monthly payments for thirty years. Since the payments are monthly, $m = 12$ and $p = \frac{7.5}{100 \cdot 12} = .00625$. Since the term is $y = 30$ years, we have $T = m \cdot y = 12 \cdot 30 = 360$. Thus

$$D = B\left(\frac{p}{1-(1+p)^{-T}}\right) = 100000\left(\frac{.00625}{1-(1+.00625)^{-360}}\right) = \$699.2145093$$

and my monthly payment will be \$699.21.

If I think I can budget \$800 a month for my mortgage payment, then I can repay a mortgage with a balance of

$$B = D\left(\frac{1-(1+p)^{-T}}{p}\right) = \$800\left(\frac{1-(1+.00625)^{-360}}{.00625}\right) = \$114,414.1017$$

or about \$114,000 dollars.

We'll come back to consider other ways to use the Present Amortization Formula (1.8.1) in a moment but first let's formalize what we did above with a method. As always, the first two steps are the same (find m and use it to get p and T) and the third just involves plugging values into the formula.

Method for solving present or loan amortizations (1.8.3)

Step 1: Determine the periods in the problem (that is, the units in which the term is measured) and the value of m, the number of periods per year.

Step 2: Use the Interest Rate Conversion Formula (1.1.11) to find the periodic interest rate p from the nominal interest rate r and the Term Conversion Formula (1.1.14) to find the term T *in periods* from the term in years y.

Step 3: Apply the appropriate PRESENT AMORTIZATION FORMULA (1.8.1) to find whichever of the the deposit D and the balance B is to be determined.

Here are a few exercises for you to try which involve mortgages and other consumer loans. Since such loans are almost invariably paid on a monthly basis, I have *not* explicitly stated the payment frequency unless it is *other* than monthly. As usual, we want to work to the nearest cent. In loans, the rounding error of a fraction of a cent per payment gets multiplied by a number of payments which can be large — in the hundreds — and this can cause an discrepancy of a few dollars between the exact value of all the rounded payments and the balance of the loan. We'll just ignore this but — surprise, surprise — the bank doesn't. The final payment is usually adjusted to makes things balance exactly. I have worked a few of the exercises as further examples.

PROBLEM (1.8.4) What is the monthly payment on a mortgage with a a balance of \$100,000 if
a) the nominal interest rate is 7.5% and the term of the mortgage is
i) 25 years?

Solution

Step 1: The periods are months so $m = 12$.

Step 2: $p = \dfrac{r}{100m} = \dfrac{7.5}{100 \cdot 12} = .00625$ and $T = my = 12 \cdot 25 = 300$.

Step 3: Here we know the loan balance $B = \$100000$ so we plug into find the payment D

$$D = B\left(\frac{p}{1-(1+p)^{-T}}\right) = 100,000\left(\frac{.00625}{1-(1+.00625)^{-300}}\right) = \$738.99.$$

ii) 20 years?
ii) 15 years?

b) the nominal interest rate is 9% and the term of the mortgage is

- i) 30 years?
- ii) 20 years?
- iii) 15 years?

Solution

Step 1: The periods are months so $m = 12$.

Step 2: $p = \dfrac{r}{100m} = \dfrac{9}{100 \cdot 12} = .0075$ and $T = my = 12 \cdot 15 = 180$.

Step 3: Here we know the loan balance $B = \$100,000$ so we plug into find the payment D

$$D = B\left(\frac{p}{1-(1+p)^{-T}}\right) = 100,000\left(\frac{.0075}{1-(1+.0075)^{-180}}\right) = \$1,014.27.$$

Problem (1.8.5) A car dealer offers to sell you a new car, “No money down and easy monthly payments”. You calculate that you can afford to make car payments of $250 a month. How much can you afford to pay for the car if

a) the nominal interest rate on the loan is 4.5% and the term is

- i) 2 years?

Solution

Step 1: The periods are months so $m = 12$.

Step 2: $p = \dfrac{r}{100m} = \dfrac{4.5}{100 \cdot 12} = .00375$ and $T = my = 12 \cdot 2 = 24$.

Step 3: Here we know the payment $D = \$250$ so we plug into find the loan balance B

$$B = D\left(\frac{1-(1+p)^{-T}}{p}\right) = \$250\left(\frac{1-(1+.00375)^{-24}}{.00375}\right) = \$5,727.66$$

ii) 3 years?
ii) 4 years?

b) the nominal interest rate is 12% and the term
i) 2 years?
ii) 3 years?
iii) 4 years?

Solution

Step 1: The periods are months so $m = 12$.

Step 2: $p = \dfrac{r}{100m} = \dfrac{12}{100 \cdot 12} = .01$ and $T = my = 12 \cdot 4 = 48$.

Step 3: Here we know the payment $D = \$250$ so we plug into find the loan balance B

$$B = D\left(\frac{1-(1+p)^{-T}}{p}\right) = \$250\left(\frac{1-(1+.01)^{-48}}{.01}\right) = \$9,493.49$$

I hope you can guess what's coming next. Yes, I'm going to ask how we can check such calculations. This time there will be a version of the CONTINUOUS APPROXIMATION (1.3.11) and of the SIMPLE INTEREST APPROXIMATION (1.3.3) and something new. To use the CONTINUOUS APPROXIMATION (1.3.11), we simply approximate the exponential $(1+p)^{-T}$ in the PRESENT AMORTIZATION FORMULA (1.8.1) by the slightly smaller exponential $e^{-(p \cdot T)} = e^{-(r \cdot y)}$ getting a slightly larger numerator. In the formula for D where this appears in the denominator and we are now *dividing* by a larger quantity, we get an approximation slightly smaller than the exact value.

PRESENT AMORTIZATION: CONTINUOUS APPROXIMATION (1.8.6) B is a bit less than $D\left(\dfrac{1-e^{-(r \cdot y)}}{p}\right)$.

As with the FUTURE AMORTIZATION: CONTINUOUS APPROXIMATION (1.6.14), there's still a periodic rate p in the formula. If you forgot to convert the nominal rate when using the PRESENT AMORTIZATION FORMULA (1.8.1), you'll probably use r for p here too. Fortunately, the different numerator will lead to a different answer and let you catch your mistake. This is another formula which you don't really need to learn. You can make the necessary approximations if you just remember to use the CONTINUOUS APPROXIMATION (1.3.11) to replace the $(1+p)^{-T}$.

EXAMPLE (1.8.7) Let's check the calculation in EXAMPLE (1.8.2). Here we had $r = 7.5\%$ and $y = 30$ years and since we were compounding monthly $m = 12$ and $p = .00625$. Our D was \$699.21 so the balance B of \$100,000 should be a bit less than

$$D\left(\frac{e^{-(r\cdot y)}-1}{p}\right) = 699.21\left(\frac{e^{-(.075\cdot 30)}-1}{.00625}\right) = \$100,082.2093$$

and it is. You can check a loan balance calculation the same way.

PROBLEM (1.8.8) Use the PRESENT AMORTIZATION: CONTINUOUS APPROXIMATION (1.8.6) to check your answers to part b) of each of PROBLEM (1.8.4) and PROBLEM (1.8.5).

To apply the SIMPLE INTEREST APPROXIMATION (1.3.3) to loans, we use the same basic idea as we did for future amortizations. However, we apply it to the balance rather than the deposits because the arithmetic is then easier. We ask, "What's the average outstanding balance on the loan over its term?" and we answer "About half the initial balance". So the simple interest on the balance should be roughly $\frac{r}{100}\cdot y\cdot\frac{B}{2}$. The total of all the payments T of $\$D$ each should match the original balance plus this interest. In fact, they should amount to somewhat more as the actual compound interest on the outstanding balance will be greater than the simple interest approximation we have used.

PRESENT AMORTIZATION: SIMPLE INTEREST APPROXIMATION (1.8.9) $T\cdot D$ is larger — possibly quite a bit larger — than $B\left(1+\frac{r}{100}\cdot\frac{y}{2}\right)$.

Once again, the best feature of this check is that you can do it roughly in your head. Here are a couple of examples.

Example (1.8.10) Let's check the calculation in Example (1.8.2). Here we had $r = 7.5\%$ and $y = 30$ years and since we were compounding monthly $m = 12$ and $T = 360$. Our D was \$699.21 so $T \cdot D = \$251,715.60$. This should be somewhat larger than $B\left(1+\frac{r}{100}\cdot\frac{y}{2}\right) = \$100,000\left(1+\frac{7.5}{100}\cdot\frac{30}{2}\right) = \$212,500.00$ and it is. As usual, the agreement is not very good here because the term was fairly long.

We'll get better agreement if we check a problem with a shorter term like a)i) of Problem (1.8.5). Here we had $r = 4.5\%$, $y = 2$, $m = 12$,$T = 24$, $D = \$250$ and $B = \$5,727.66$. So we expect $T \cdot D = \$6,000$ to be a bit larger than $B\left(1+\frac{r}{100}\cdot\frac{y}{2}\right) = \$5,727.66\left(1+\frac{4.5}{100}\cdot\frac{2}{2}\right) = \$5,985.40$. Since the term was short we get excellent agreement.

Problem (1.8.11) Use the Present Amortization: Simple Interest Approximation (1.8.9) to check your answers to part b) of each of Problem (1.8.4) and Problem (1.8.5).

One nice feature of loan amortizations is that when the term is long — exactly when Present Amortization: Simple Interest Approximation (1.8.9) is way off — there is another and even easier way to check your answer. The idea is that when T is big the negative exponential $(1+p)^{-T}$ will be small. For example in Example (1.8.2). we have $T = 360$ and $p = .00625$ so $(1+p)^{-T} = .1061398302$. So we don't lose too much by ignoring this. We call this the Interest Approximation (1.8.12) because the estimate it give for the payment D simply equals one periods *interest* on the initial balance of the loan. If you made this payment every month, your outstanding balance would never change: each month you'd pay off the preceding month's interest but you'd never reduce the outstanding balance on the loan. So you'd be paying the loan forever. One consequence is that this approximation would be exactly correct if the term were infinite and is accurate only when the term is fairly long. We get an estimate for B which is bit large and one for D which is a bit small and both are stunningly simple.

INTEREST APPROXIMATION (1.8.12) If an amortized loan has a long term, the the balance B is slightly smaller than $\dfrac{D}{p}$ and the payment D is slightly larger than $B \cdot p$.

EXAMPLE (1.8.13) Let's check the calculation in EXAMPLE (1.8.2). Here we had $p = .00625\%$ and $B = 100,000$ so we'd expect D which turned out to be $\$699.21$ to be a bit larger than $100,000 \cdot .00625 = \$625$. Note how simple the check is.

The downside here is that for loans with short terms the approximation is very poor. For part a) i) of PROBLEM (1.8.5), where $p = .00375$ and $B = \$5,727.66$ we get the estimate $D \cong \$5,727.66 \cdot .00375 = \$21.48$ which is *much* smaller than the exact payment of $\$250$.

PROBLEM (1.8.14) Compute the monthly payment on a $80,000 mortgage at a rate of 8.4% and compare it with the INTEREST APPROXIMATION (1.8.12) for terms of
a) 20 years.
b) 30 years.
c) 40 years.