3 Approximating compound interest

The SIMPLE INTEREST FORMULA (1.1.6) is just that: simple. As long as you remember the PERIOD WARNING (1.1.13) and INTEREST RATE WARNING (1.1.9) and you just multiply and it is pretty hard to go wrong. What's more there is an easy way to check any answer: since changing the periods used to measure the term has no effect on the final value of the interest, you can just recalculate with different periods. The COMPOUND INTEREST FORMULA (1.2.4) is not really more complicated. However, since the choice of the compounding period now has a substantial effect on the answer, you can't just blindly recalculate with different periods to check. In this section, we'll learn how to keep our eyes open while changing periods. This leads to several easy approximations for compound interest which can be used to check that a compound interest answer at least "looks right". While doing this, we'll find out that calculators don't always compute the compound interest formula correctly and learn how to work around their limitations. In later sections (YIELDS (1.4) and INFLATION, POPULATION, COMPUTATION, RADIATION (1.5), for example), what we'll learn will turn out to be handy in many other ways.

The simple interest approximation

The first approximation we'll use is truly simple: just ignore compounding and use the SIMPLE INTEREST FORMULA (1.1.6). If we use years as periods — so $p = \frac{r}{100}$ and $T = y$ — then the interest on an amount A_0 is $I = p \cdot A_0 \cdot T = \frac{r}{100} \cdot A_0 \cdot y$ and the future value A_T — the total of principal plus interest at the end of the period — is *roughly* $A_0 + \frac{r}{100} \cdot A_0 \cdot y = A_0(1 + \frac{r}{100} \cdot y)$. We can use this to approximate the compound interest formula.

Here's an exanple.

Example (1.3.1) Let's try checking my answer to one of the Problems in Compound interest (1.2). In part b) of Problem (1.2.10) which asked for the future value S of \$2,600 at nominal interest of 9% for a term of 3 years compounded quarterly, my answer was $\$3,395.73$. The Simple Interest Approximation (1.3.3) gives $A_0(1 + \frac{r}{100} \cdot y) = \$2600(1 + .09 \cdot 3) = \$3,302.00$ which is just a bit smaller than my answer.

If we think of A_0 as an initial amount or present value B and think of A_T as a final amount or future value S, we can rewrite the approximation

$$A_T \cong A_0(1 + \frac{r}{100} \cdot y) \quad \text{as} \quad S \cong B(1 + \frac{r}{100} \cdot |y|)$$

This new formula on the right can be used to check both future and present value problems. Just remember that S is to be the final amount and B the starting amount, so in a present value problem we'd have to reverse the A's: $S = A_0$ and $B = A_T$. The bars around the y are absolute value signs and remind us to *always* make y positive in this formula — just as we have always done when working with simple interest.

Example (1.3.2) In part a) of Problem (1.2.16) which asked for the present value of a sum of $S = \$4200$ which will be due in 4 years at a nominal interest rate of 12% compounded annually, I got $B = \$2669.18$. The Simple Interest Approximation (1.3.3) gives $B(1 + \frac{r}{100} \cdot |y|) = \$2669.18(1 + .12 \cdot 4) = \$3950.39$. There are two points to note. First, this time we plugged in the *answer* as the present value or starting amount — because we were checking a present value problem. Second, even though I used a negative period of -4 years in Problem (1.2.16) I plugged in $|y| = 4$ in the check.

Unfortunately, this "approximation" is only close to the correct value when the term is quite short. If you look over the example at the start of section Compound interest (1.2), you'll see that over terms of many years, the interest on the interest in a compound interest calculation can become much larger than the original amount or the interest on the original amount. However, we can still make some use of this formula by thinking of it somewhat differently. Using the Simple Interest Formula (1.1.6) amounts to making the entire term a single period so that, in effect, there is no compounding. In other words we want $T = 1$. We can check this idea by working work backwards. Since $T = m \cdot y$ by the Term Conversion Formula (1.1.14), $m \cdot y = 1$ so $m = \frac{1}{y}$. Then, by the Interest Rate Conversion Formula (1.1.11)

$$p = \frac{r}{100 \cdot m} = \frac{r}{100 \cdot \frac{1}{y}} = \frac{r}{100} \cdot y\,.$$

Finally, applying the Compound Interest Formula (1.2.4)

$$A_T = A_0(1+p)^T = A_0(1 + \frac{r}{100} \cdot y)^1 = A_0(1 + \frac{r}{100} \cdot y)\,.$$

By combining this with what we know about the Effect of More Frequent Compounding (1.2.19), we can squeeze out a bit of information even when the approximation is way off. Remember that more frequent compounding increases future values and decrease present values. Since using simple interest amount to doing the least compounding possible — none at all — it should definitely underestimate future values. Note that both examples above confirm this: the approximations are both slightly smaller than the answers being checked.

Simple Interest Approximation (1.3.3) The ending or future amount S in a compound interest calculation should be greater than the approximation $B(1 + \frac{r}{100} \cdot |y|)$ (where B is the starting or present value) and the approximation should be fairly good if the term is not too long.

Both EXAMPLE (1.3.2) and EXAMPLE (1.3.2) illustrate reasonable uses of the SIMPLE INTEREST APPROXIMATION (1.3.3). I often do something even cruder in my head when working problems in class. In the first case, I'll say: "The interest is $.09 \cdot 3 = .27$ which is about $\frac{1}{4}$ and $\frac{1}{4}$ of \$2600 is \$650 so I should expect my answer to be a bit bigger than \$3250". Similarly, in the second problem, I'd say, "Here the interest is $.12 \cdot 4 = .48$ which is about $\frac{1}{2}$ so I expect \$2669.18 times $(1 + \frac{1}{2}) = \frac{3}{2}$ to be a bit smaller then \$4200; it's easier to turn this around $\frac{2}{3}$ of \$4200 or \$2800 should be somewhat bigger than \$2669.18". The point is that since we are only approximating anyway we can afford to ignore the difference between .27 and $\frac{1}{4}$ or between .48 and $\frac{1}{2}$. I know I am much more likely to perform a quick mental check than one where I have to get out my calculator and a check you don't perform is not much of a check.

EXAMPLE (1.3.4) To see the limitations of the approximation, let's look at part c) of PROBLEM (1.2.17) which asked for the present value of an amount of \$100,000 due in 40 years at an interest rate of 4.8% compounded monthly. My answer was \$14,716.95. The SIMPLE INTEREST APPROXIMATION (1.3.3) gives $B(1 + \frac{r}{100} \cdot y) = \$14,716.95(1 + .048 \cdot 40) = \$42.973.49$. Again this is less than the correct futue value of \$100,000, but now it is so much less — barely two-fifths — that is not much use as a check. The period here was just too long for the approximation to be useful.

SELF-TEST

PROBLEM (1.3.5) Use the SIMPLE INTEREST APPROXIMATION (1.3.3) to check your answers to PROBLEM (1.2.11), PROBLEM (1.2.21) and PROBLEM (1.2.23). Are there any other problems in COMPOUND INTEREST (1.2), for which you'd expect the SIMPLE INTEREST APPROXIMATION (1.3.3) to be fairly accurate?

The continuous approximation

We have already seen in EFFECT OF MORE FREQUENT COMPOUNDING (1.2.19) that keeping the nominal rate and the term in years fixed, the more often we compound the larger the amount owed at the end of the term. Let's go back to the PROBLEM (1.2.2) where we borrowed $100,000 at 8% interest and ask: What happens if the bank compounds ever more frequently? Let's try compounding monthly, daily, hourly and once a second in the two problems above. The corresponding vales of m are : 12, 365, 8760 and 31,536,000. (In other words, there are 8760 hours and 31,536,000 seconds in a year. So while a million seconds might seem like forever it is actually only about 11.57 days.) Now we can just apply the METHOD FOR FINDING COMPOUND INTEREST (1.2.15) —- I have omitted the calculations — to get TABLE (1.3.6) below for a term of 3 years:

m	12	365	8760	31,536,000
T	36	1095	26280	94,608,000
p	$0.00\dot{6}$	.2191780822e-3	.9132420088e-5	.2536783358e-8
A_T	$127,023.71	$127,121.56	$127,123.37	$132,819.91

TABLE (1.3.6) COMPOUNDING TABLE FOR A 3 YEAR TERM

and TABLE (1.3.7) for a term of 12 years.

m	12	365	8760	31,536,000
T	144	4380	105120	378,432,000
p	$0.00\dot{6}$	.2191780822e-3	.9132420088e-5	.2536783358e-8
A_T	\$260,338.92	\$261,142.08	\$261,156.97	\$311,209.46

Table (1.3.7) Compounding table for a 12 year term

Problem (1.3.8) Make your own calculation of each of the amounts in the tables above. You should get the answers in the table to the penny when you compound monthly. Some of your other answers may be somewhat different for reasons I'll explain. If so, don't worry.

There are many interesting things to note about these tables. Let's look at the p rows to start. First notice that these rows in the two tables contain identical values: we should expect this since p does not depend on the *term* of the loan — which is what differs between the two tables — but only on the *frequency* with which we compound. Second, notice how tiny the periodic rates have become: this is because we are compounding many times a year with very short compounding periods and hence we get very little interest in each period. The values are so small that my calculator has given them to me in scientific notation so I will have as many decimals as possible. Recall that a value like $.2536783358e-8$ stands for $.2536783358 \times 10^{-8} = 0.000000002536783358$ and my calculator screen just does not have room for all those 0's. Notice also that I did not round this number to preserve the accuracy of the final answer.

Next, let's look at the final amount of A_T rows. Does anything strike you about these? All the amounts are getting closer and closer together — in the three year table they seem to be settling down around $127,120 or so and in the 12 year table around $261,150 — and then suddenly the last amount where we compound in seconds is much bigger. What's going on? Two things. First, the final amounts are *wrong*! The problem is that to find them I asked my calculator to compute

$$A_T = 100000 * (1 + .2536783358e - 8)^{94,608,000} \quad \text{and} \quad A_T = 100000 * (1 + .2536783358e - 8)^{378,432,000}$$

and those exponents made it choke. It just can't compute powers that big accurately. The 10 digits of precision it uses is just not enough to get the final amount even to 2 places!! If I use a much better calculator (which carries 20 places) and make the same calculation, I find that the amounts turn out to be

A_T, 3 years	\$127,023.71	\$127,121.56	\$127,124.78	\$127,124.92
A_T, 12 years	\$260,338.92	\$261,142.18	\$261,168.50	\$261,169.65

Table (1.3.9) Frequent compounding table computed with a 20 place calculator

Notice that it turns out it wasn't just the final "seconds" amounts that were wrong. They were just the only wrong answers that were so far off that it was clear to the naked eye that something was fishy. Both the amounts for hourly compounding were off by dollars and the 12 year final amount with daily compounding — which is what your bank uses — was off by 10 cents. But all these other answers were close enough that the only way we would ever know that they were wrong was by making a second more accurate calculation. The moral here is:

Murphy's Law of Calculators (1.3.10) Never trust a calculator's answers unless you have some other way to check them.

In particular, we have now learned: *Never use a calculator to take a power with a very large exponent.* You may be wondering how you are going to do problems like the one where we got the wrong answer if you are not able to use your calculator. Relax: I'm just not going to ask *you* to work any problem where the exponent in the COMPOUND INTEREST FORMULA (1.2.4) is dangerously large. But, in real life where daily compounding is common, you might be asked to: if you are, remember to watch out.

Did some of the amounts you computed in PROBLEM (1.3.8) differed from mine as I suggested in the problem that they might? Perhaps now you can guess why I told you not to worry about this. I knew my answers were wrong and (unless you have a very good calculator) that yours probably would be wrong too. But there's no chance we'd get the *same* wrong answers. Why? Because every calculator is a bit different inside. While they'll all give the same answer when they can get the right one, when things go wrong each calculator goes wrong in its own way.

So much for the bad news. Let's get back to those amounts. The correct answers give a striking confirmation of our initial impression that as you compound more and more frequently, the final amounts get larger and larger but do so ever more slowly. Eventually, these amounts appear to settle down. In fact, no matter how often you compound — even if you compound a trillion times a second — the final amount you'll wind up with in these two problems will never grow by another *cent*: after 3 years, you'll have \$127,124.92 and after 12 you'll have \$261,169.65. (You need a calculator that keeps 25 places to check these answers so you'll just have to trust me on this. If this worries you a bit, take a gold star: you're catching on.)

What would be very nice is to have some "easy way" to get this magic maximum amount. It's not that we'd ever want to compound interest every second in real life and so have a direct need for a way around the limitations of our calculators. But, if we did we'd have a good way to make a rough check of any compound interest calculation. Our answer should be close to, but somewhat smaller than this magic amount: the more frequently we compound, the closer the two should be. In particular when we compound monthly or daily as in the majority of real world loans we should see a few matching digits, as in the tables above. If we do not, we'll know right away that something is wrong.

Well kids, life is good! Finding formulas for limits – which is fancy way to say, for how things "settle down" — is one of the main applications of calculus. And, a standard formula from calculus computes the magic amount which appears in the tables above. Moreover, you don't need to know any calculus to understand and use this formula.

CONTINUOUS APPROXIMATION (1.3.11) A_T lies between A_0 and the *continuous approximation*

$$A_0 \cdot e^{(p \cdot T)} = A_0 \cdot e^{\left(\frac{r}{100} \cdot y\right)} .$$

If the compounding period is short, A_T is close to the continuous approximation.

THE NUMBER e (1.3.12) The base e in this formula is a very important number: $e \cong 2.7182818284590452353 6.$ But you don't ned to try to memorize any of these decimals: e is so important it's got its very own key on your calculator. Moreover, exponentials with base e occur so often in so many places that there is also a key usually called **exp** for taking the exponential base e of the current value.

So to use the CONTINUOUS APPROXIMATION (1.3.11), you just calculate the product $\frac{r}{100} \cdot y$, hit **exp** and multiply by the amount A and you've got the magic amount. In the two problems above, it gives

$$A \cdot e^{\left(\frac{r}{100} \cdot y\right)} = \$100000 \cdot e^{.08 \cdot 3} = \$127,124.92 \quad \text{and} \quad A \cdot e^{\left(\frac{r}{100} \cdot y\right)} = \$100000 \cdot e^{.08 \cdot 12} = \$261,169.65 .$$

Try it with your calculator!

You might wonder why I preferred to calculate the exponent in the form $\frac{r}{100} \cdot y$ rather than in the apparently simpler form $p \cdot T$. First, let's remark that they really are equal. Using the INTEREST RATE CONVERSION FORMULA (1.1.11) and TERM CONVERSION FORMULA (1.1.14) gives $p \cdot T = \frac{r}{100 \cdot m} m \cdot y = \frac{r}{100} \cdot y$. Forgetting to use one of these conversion formulas is the most common error in working interest problems. Thus, the fact that I can use the nominal interest rate and the term in years in the formula with no need for converting to periodic rates and periods, makes the continuous approximation perfect for catching such errors. It's one of the nicest features of the formula that it lets us work with the real life quantities we like to think in terms of — nominal rates and years.

But the formula has other amazing properties. First, even with a standard calculator you can use it to find the magic number to the penny. Using the COMPOUND INTEREST FORMULA (1.2.4), I needed a supercalculator to get these numbers. You'd have no way to compute them. Makes you think there might be something to this calculus after all, and that it might not be as hard as it's cracked up to be. (Both guesses are correct and I hope this section will inspire a few of you to take a calculus course. If you are planning to do serious work in any of the mathematical, computational, physical, biological or social sciences, you will have to do so eventually and the sooner you start the easier a time you'll have with the math and the further ahead you'll be in your major. If you don't believe me, ask your major's undergraduate advisor.)

The final remarkable feature of the CONTINUOUS APPROXIMATION (1.3.11) is that the signs of the quantities T and y which measure time are not mentioned anywhere. So far we have only used the formula in future value problems where these quantities are positive but everything works just as well in present value problems when they are negative. The future value version says that when T and y are positive that $A_0 < A_T < A_0 \, e^{\left(\frac{r}{100} \cdot y\right)}$ or, since $A_T = A_0\,(1+p)^T$ that

$$A_0 < A_0\,(1+p)^T < A_0\,e^{\left(\frac{r}{100}\cdot y\right)}$$

. The present value version says that if we replace T by $-T$ and y by $-y$ we should have

$$A_0 > A_0\,(1+p)^{-T} > A_0\,e^{\left(\frac{r}{100}\cdot(-y)\right)}\,.$$

Note that, although the directions of the inequalities are reversed, the exact value A_T is still in the between A_0 and the continuous approximation as claimed. I leave this to you: it is good practice in playing with exponents and inequalities: you need no special knowledge about the number e.

PROBLEM (1.3.13) Show that if $A_0 < A_0\,(1+p)^T < A_0\,e^{\left(\frac{r}{100}\cdot y\right)}$ then $A_0 > A_0\,(1+p)^{-T} > A_0\,e^{\left(\frac{r}{100}\cdot(-y)\right)}$.

If you are hoping that I will now explain where this approximation comes from, bless you. First off, there's not much doubt that the CONTINUOUS APPROXIMATION (1.3.11) is correct. The fact that it computes the two amounts above to the penny is pretty convincing. And, as we've already noted you don't need to understand where it comes from to use it. So if you could care less, you can skip to EXAMPLE (1.3.15).

Everything comes down to a set of approximations to the number e: if N is a large positive number, then $\left(1+\frac{1}{N}\right)^N$ is slightly smaller than e. The bigger N you take, the closer the approximation. You can convince yourself of this by calculating with a few big values of N. For example,

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.716923932 \qquad \text{and} \qquad \left(1+\frac{1}{10000}\right)^{10000} = 2.718145927\,.$$

The first less than e by about .013 and the second by about .0014.

PROBLEM (1.3.14) You shouldn't take N too big, however. Why? What will happen if you do?

I'm afraid if you want to know where this formula comes from you'll have to take that calculus course. But we can easily see how it leads to the continuous approximation. If we take the $(p \cdot T)^{\text{th}}$ power of both sides, we find that

$$\left(\left(1+\frac{1}{N}\right)^{N}\right)^{(p\cdot T)} = \left(1+\frac{1}{N}\right)^{(N\cdot p\cdot T)} \quad \text{is a bit smaller than} \quad e^{(p\cdot T)}.$$

The CONTINUOUS APPROXIMATION (1.3.11) just says we can always find a value of N for which the exponential $\left(1+\frac{1}{N}\right)^{(N\cdot p\cdot T)}$ equals the exponential $(1+p)^T$ in the COMPOUND INTEREST FORMULA (1.2.4). Equating bases we need $\frac{1}{N} = p$ and equating exponents we need $N \cdot p \cdot T = T$: there are two equations for the single unknown N which would usually be impossible to satisfy. However, here is where a small miracle happens. Solving the first equation tells that we must have $N = \frac{1}{p}$. If we plug this into the left side of the second equation it becomes $N \cdot p \cdot T = \frac{1}{p} \cdot p \cdot T = T$ so the exponents automatically match up too! The final point to note is that to get a good approximation we need to have N large. But

$$N = \frac{1}{p} = \frac{1}{\frac{r}{100\cdot m}} = \frac{100 \cdot m}{r}.$$

Thus N is big when m is: in other words, we get a good approximation when we are compounding frequently.

Any way, using this approximation is a cinch. Let's use it to check a few problems from the last section.

EXAMPLE (1.3.15) PROBLEM (1.2.11) asked about an amount of \$1,255,000 earning nominal interest of 6.73%

for a term of 5 years. In part c), where we compounded monthly I got a future value of \$1,755,397.74. The Continuous Approximation (1.3.11) gives $\$1,255,000e^{\left(\frac{6.73}{100}\cdot 5\right)} = \$1,757,048.78$ which is, as predicted, slightly higher but matches the exact answer to 3 places.

Example (1.3.16) In part b) of Problem (1.2.18), we computed the present value of \$100,000 due in 20 years at interest of 9.6% compounded quarterly to be \$14,996.97. The Continuous Approximation (1.3.11) gives $\$100,000e^{\left(\frac{9.6}{100}\cdot(-20)\right)} = \$14,660.70$. Note that since this was a present value problem where we were moving money backwards in time, we used a negative value $y = -20$ and that this time, as expected, the continuous approximation is slightly lower than the exact answer. Because we are compounding less frequently here, we get a less accurate approximation — only 2 places match. But, the approximation is good enough that we'd be sure to catch any silly errors like forgetting to convert from rate or term or miskeying one of the numbers into our calculator.

Self-Test

Problem (1.3.17) Use the continuous approximation to check your answers to the 4 problems Problem (1.2.20) to Problem (1.2.23).