# GRÖBNER TECHNIQUES FOR LOW DEGREE HILBERT STABILITY

IAN MORRISON AND DAVID SWINARSKI

## 1. Introduction

We analyse the Hilbert stability of bicanonical models of certain curves $X$ of small genus with suitably large automorphism groups with respect to linearizations of fixed small degree $m$. Our examples are very special but they have geometrically interesting applications discussed below.

Our analysis has two main novelties. First, we give a method for deducing the stability, always with respect to $\mathrm{SL}(V)$), of the Hilbert point of a subscheme $X$ of $\mathbb{P}(V)$, from a symbolic calculation of certain state polytopes. Even the possibility of such a reduction for Hilbert points of subschemes of large codimension is new and the hypotheses on $\mathrm{Aut}(X)$ enter into it an essential way.

Second, existing approaches such as, most notably, those pioneered by Gieseker in [11–13] have an asymptotic character and verify Hilbert stability only with respect to linearizations of sufficiently large degree $m$. Our method allows us to verify stability with respect to linearizations of fixed degree $m$. The values of $m$ in our main examples are not merely fixed but quite small, typically 6 or less.

The bicanonical curves $X$ and small degrees $m$ in our main examples are chosen because quotients of loci in the bicanonical Hilbert schemes $\mathbb{H}$ in question are predicted to yield new log minimal models of the moduli spaces of stable curves. For further details on this connection, see 7.5 of [22] and the references cited there. In addition to providing applications of our examples, the log minimal model program makes, by very indirect arguments, very specific predictions about the degrees in which their bicanonical Hilbert points will be stable, strictly semistable and unstable. Our examples verify these predictions exactly, providing further evidence for them and a useful check on our calculations. We postpone giving further details until we discuss our calculations in Section 7.

A disclaimer is in order here. Our results here show the non-emptiness of stable loci of interest in the log minimal model program in small genus but are far from producing the desired quotients. Therefore, we do not discuss these quotients further here—they will be the subject of a future paper—and deal only with our methods for checking stability and our examples.

Our key assumption on $X$ is that it is *bicanonically multiplicity free* (4.11): the multiplicity, in the natural representation of $\mathrm{Aut}(X)$ on $V = H^0\big(X, K_X^{\otimes 2}\big)$, of every irreducible representation is either 0 or 1. The bicanonically multiplicity free curves that we use as examples are certain special hyperelliptic curves called Wiman curves $\mathcal{W}_g$ that are well known in the literature on curves with automorphisms [6], and nodal curves that are joins of 2 or more Wiman curves.

Our approach combines this hypothesis with theorems of Kempf on worst destabilizing 1-ps's to reduce checking stability for the full group $\mathrm{SL}(V)$ to checking stability with respect to a distinguished maximal torus $T$ (Corollary refslhilbertstabilityfroompolytope). There is an easy naive algorithm for checking this symbolically but its complexity makes it impractical except in very simplest cases. By adapting results of Bayer and the first author on state polytopes, we give, in Corollary 3.13, an algorithm efficient enough that we are able to handle examples arising in our intended applications. The calculations are carried out in

*Date*: Authors' working draft of February 7, 2009.

2000 *Mathematics Subject Classification.* Primary 14L24, 14H10
Secondary 14D22, 13P10.

*Key words and phrases.* Hilbert stability, state polytope.

Macaulay2 [30] using the statePolytope package of the second author which calls the packages gfan [29] and polymake [10,32] to compute intermediate results. Detailed output from our calculations and the source code of our routines are available at the second author's webpage.

Working with small degree $m$ is a sword that cuts both ways. On the one hand, the $m$ we work with are well below the bounds that ensure various standard uniformity hypotheses for ideals of points of $\mathbb{H}$, even those that are deformations of smooth subschemes. A typical example is that the degree $m$ graded pieces of the homogenous ideals do not yield the embedding of $\mathbb{H}$ as a closed subscheme of a Grassmanian that is used to linearize the relevant PGL-action. Here we have addressed these complications by replacing $\mathbb{H}$ with a multigraded Hilbert scheme $\widehat{\mathbb{H}}$ in the sense of Haiman and Sturmfels [14].

On the other hand, even the more efficient algorithm we use is only practical for computing state polytopes in fairly low degrees. It involves computing *all* the monomial initial ideals $X$ (in the coordinates giving the special torus $T$) and requires a Gröbner basis calculation for *each* initial ideal. In fact, as the genus of $X$—and hence the bicanonical embedding dimension—increased, we were often unable to carry even these low degree calculations to completion because there are simply too many such ideals. To understand such examples, we use several additional, somewhat ad-hoc tricks. The first involves a Monte Carlo strategy that computes a random sub-polytope of the state polytope by computing some random initial ideals. If $X$ is Hilbert stable and we are fortunate, this sub-polytope provides a proof of stability. This approach can never prove that $X$ is unstable but we are able to do this, when necessary, by educated guesswork. To show that $X$ is unstable, it suffices to find a single destabilizing one-parameter subgroup $\lambda$. Geometry—in our examples, analogies with completed stages of the log minimal model program—often suggests what this $\lambda$ should be. Given the right $\lambda$, a *single* Gröbner basis computation suffices to check that $\lambda$ is destabilizing. Finally, the Parabola Trick (Proposition 5.2) uses ideas of Hassett, Hyeon and Lee [18] to deduce stability of Hilbert points of smooth curves in low degrees not accessible to our calculations from their stability in even lower degrees.

Here is a summary of the plan of the rest of the paper. The details of our multigraded setup for Hilbert points is given in Section 2 and of the results on state polytopes we need are extended to this setting in Section 3. Section 4 reviews Kempf's results on worst one-parameter subgroups and explains how, for multiplicity free $X$, they reduce checking stability to calculations with state polytopes. The Monte Carlo version and the Parabola Trick are outlined in section 5. Section 6 recalls facts about Wiman curves and pluricanonical equations of hyperelliptic curves needed to set up the Macaulay2 calculations for our examples. These calculations, what they say about stability of the bicanonical models and how these results it with the predictions of the log minimal model program are reviewed in Section 7. Finally, we close by listing some ideas for future work.

**Acknowledgements.** It is a great pleasure to thank Brendan Hassett, David Hyeon, and Yongnam Lee, whose work got us interested in these GIT questions, and Dave Bayer, Johan de Jong, Bill Graham, Anders Jensen, Julius Ross, Greg Smith, Mike Stillman and Bernd Sturmfels for helpful discussions. The first author wishes to acknowledge support from a Fordham University Faculty Fellowship during the early stages of this project. The second thanks Sonja Mapes for her expert instruction on programming in Macaulay2, and Dan Grayson for suggesting many improvements to the statePolytope package.

## 2. Hilbert points and state polytopes

**Parameter schemes adapted to low degrees.** Fix an $(N+1)$-dimensional vector space $V$ and a set of coordinates $\{x_0, \ldots, x_N\}$ identifying $V$ with $K^{N+1}$ and the homogeneous coordinate ring of $\mathbb{P}(V)$ with $S := K[x_0, \ldots, x_N]$. Fix also a Hilbert polynomial $P$ of degree $r$ and let $\mathbb{H}$ be the Hilbert scheme of subschemes $X \subset \mathbb{P}(V)$ with Hilbert polynomial $P$

The goal of this section is to define state polytopes of such subschemes $X$—or of their homogeneous ideals $I \subset S$—and to recall their connection to the stability of the Hilbert point of $X$ with respect to the action of $\mathrm{PGL}(V)$ induced by the natural action on $\mathbb{P}(V)$. Both of these notions depend on the choice of the degree $m$ that is used to linearize this action. To make uniform sense of either the Hilbert point or the state polytope for *all* $X$ having a fixed Hilbert polynomial $P$—that is, over the whole of $\mathbb{H}$—it is necessary to take $m \geq m_P$ for a sufficiently large $m_P$. (For instance, we can use the Gotzmann number $m_0$ for $m_P$.)

However, the applications we have in mind to stability problems arising in the log minimal model program for $\overline{M}_g$ (cf. [16, 17]) require us to work with a fixed degree $m < m_P$. The main goal of this section is to outline how to transfer the standard constructions to this setting. This is most conveniently achieved by using the multigraded Hilbert schemes constructed by Haiman and Sturmfels [14]. In doing this we have treated general $r$, since doing so entails no additional complications, but for the applications cited above, we will specialize to the case of curves. We also postpone to a future paper any discussion of what the results here imply about the corresponding GIT quotients of $\mathbb{H}$.

One more remark, before we begin, for those familiar with state polytopes. The variant we define here is based on the construction of Bayer and the first author in [4, §2] which works entirely in a fixed degree. We do not use the state polytopes of Sturmfels [25, Theorem 2.5] which are the Minkowski sum of these for all degrees up to the fixed one.

We begin with a definition of convenience.

**Definition 2.1** *An $r$-dimensional subscheme $X$ of $\mathbb{P}(V)$ with ideal sheaf $I$ is called $\ell$-*nice *if*

1. *$X$ is of pure dimension $r$.*
2. *The natural map $V^\vee \to \Gamma(X, \mathcal{O}_X(1))$ is an isomorphism.*
3. *$\mathcal{O}_X$ is $(\ell - 1)$-regular.*
4. *$I_X$ is $\ell$-regular.*

The second hypothesis may be viewed more geometrically as saying that $X$ is embedded in $\mathbb{P}(V)$ by a complete non-degenerate linear series. The third hypothesis implies that for $m \geq \ell$, the sheaf $\mathcal{O}_X(m)$ has no higher cohomology, and hence that its Hilbert polynomial $P(m)$ computes $h^0\big(X, \mathcal{O}_X(m)\big)$. Likewise, the fourth hypothesis implies that the restriction maps $S_m \to H^0\big(X, \mathcal{O}_X(m)\big)$ are surjective for all $m \geq \ell$ and that $I_X$ is generated be elements of degree at most $\ell$.

We denote by $\mathbb{H}_l$ the $l$-nice locus in the Hilbert scheme $\mathbb{H}$ of subschemes of $\mathbb{P}(V)$ with Hilbert polynomial $P$. Fix an $l$-nice subscheme $X$. We let $\widehat{R}(m) = \dim_K(S_m) = \binom{m+N}{m}$ and $\widehat{Q}_\ell(m) = \dim_K(I_m)$ for $m \geq \ell$ and $\widehat{Q}_\ell(m) = 0$ for $m < \ell$. In other words, $\widehat{Q}_\ell$ is the Hilbert function of the ideal $\widehat{I}_\ell$ given by truncating $I$ in degrees below $\ell$. As usual, we can recover $I$ from any $\widehat{I}_\ell$ by saturating. Our hypotheses imply that $\widehat{I}_\ell$ is generated in degree exactly $\ell$. Finally, let $\widehat{P}_\ell(m) = \widehat{R}(m) - \widehat{Q}_\ell(m)$. This is a truncation of the Hilbert function of $X$ and only equals $P(m)$ for $m \geq m_P$.

We denote $\widehat{\mathbb{H}}_\ell$ the multigraded Hilbert scheme of ideals in $S$ with Hilbert function $\widehat{P}_\ell$ and denote by $[I]$ the point of $\widehat{\mathbb{H}}_\ell$ determined by the ideal $I$. By [14, Corollary 1.2], $\widehat{\mathbb{H}}$ is a projective scheme representing the functor of *locally free* families of such ideals and hence is equipped with a universal family. Their Lemma 4.1 identifies $\widehat{\mathbb{H}}_{m_P}$ with the usual Hilbert scheme $\mathbb{H}$ and, if $\ell < m_P$, then truncation up to degree $m_P$ gives a map $i_\ell : \widehat{\mathbb{H}}_\ell \to \mathbb{H}$.

A few cautions are in order here. First, the $l$-nice locus in $\widehat{\mathbb{H}}_\ell$ is only locally closed, and it need not even be dense—there may be entire components of $\widehat{\mathbb{H}}_\ell$ containing no $l$-nice ideals.

Second, while $i_\ell(\widehat{\mathbb{H}}_\ell)$ is closed in $\mathbb{H}$ and is injective on the $l$-nice locus, the map $i_\ell$ need not be an embedding. This pathology has its origin in the fact that the ideals parameterized by $\widehat{\mathbb{H}}_\ell$ need not be saturated, even in degrees above $\ell$ where they are not truncated. For example, if $\mathbb{H}$ contains a point $X'$ whose (saturated) ideal $I'$ satisfies $\dim_K(I'_\ell) > \dim_K(I_\ell)$ and $\dim_K(I'_m) = \dim_K(I_m)$ all $m > \ell$, then every choice of a $\dim_K(I_\ell)$-dimensional subspace of $I'_\ell$ determines an ideal $I'' \in \widehat{\mathbb{H}}_\ell$ mapping to $X'$. Such examples can be found, for example, with $\mathbb{H}$ the Hilbert scheme of twisted cubics. The upshot is that we cannot replace the ideal $I$ parameterized by a point of $\widehat{\mathbb{H}}_\ell$ by the subscheme $X$ it determines unless we know that the degree $\ell$ truncation of the saturation of $I$ has Hilbert function exactly $\widehat{P}_\ell$, as we do, by definition, over the $l$-nice locus.

[When is the monomial initial form $\mathrm{in}_\leq(I)$ of a truncated saturated $I$ determined by a term order $\leq$ also truncated saturated? When it is, then $i_\ell(\mathrm{in}_\leq(I))$ determines the point of $\widehat{\mathbb{H}}_\ell$ parameterizing $\mathrm{in}_\leq(I)$. IM]

**The Hilbert matrix.** Henceforth we fix values of $\ell$ and $m \geq \ell$. In our applications, we often take $\ell = 2$. We begin with two remarks designed to lighten our notation. First, since $m \geq \ell$, $\widehat{P}_\ell(m)$ depends only on $m$, so we can and will omit the subscript $\ell$s used above. Second, we introduce many objects depending on our choice of $m$ in this section, but when there is no risk of confusion, we will omit the $m$ to simplify notation in later sections.

Let $W_m = \bigwedge^{\widehat{P}(m)} S_m$ and let $\mathbf{Gr}_m \subset \mathbb{P}(W)$ be the Plücker embedding of the Grassmannian $\mathbf{Gr}_m := \mathbf{Gr}\big(\widehat{P}(m), \widehat{R}(m)\big)$ of $\widehat{P}(m)$-dimensional quotient spaces of $S_m$. There is a map $g_m : \widehat{\mathbb{H}}_\ell \to \mathbf{Gr}_m$ sending $[I]$ to $S_m/I_m$. The Plücker map $g_m$ has closed image, but need not be injective: for example, $g_2$ has the same value on the the monomial ideals 3 and 4 in Example 3.12 and on the ideals 7 and 8. If, however, $I$ is generated in degrees at most $m$—in particular, for points in the $l$-nice locus—$g_m([I])$ does determine $I$.

We want to describe the homogeneous coordinates $y_A$ of $g_m([I]) \in \mathbb{P}(W)$ in a form usable in tools like `Macaulay2`. This is most conveniently and concretly done by working with the *subspace* $I_m$ of $S_m$ rather than the quotient $S_m/I_m$, and using it to define $m$-Hilbert matrices $M_{I,m}$. First let $\mathcal{B}_m = \{x^j\}$ be the monomial basis of $S_m$ with a fixed ordering. Then let $\mathcal{C}_m(I) = \{p_i, i = 1, \dots, \widehat{P}(m)\}$ be any ordered basis of $I_m$ and let $M_{I,m}$ be the $\widehat{P}(m) \times \widehat{R}(m)$ matrix whose $ij^{\text{th}}$ entry is the coefficient of the monomial $x^j$ in the equation $p_i$. The Plücker *coordinates* $y_A$ of are then simply the $\widehat{Q}(m) \times \widehat{Q}(m)$ minors of $M_{I,m}$—one for each Plücker *set* $A$ of $\widehat{Q}(m)$ of the monomials $\mathcal{B}_m$. As in the discussion on page 211 of [4], if $M'_{I,m}$ is the matrix associated to a second basis $\mathcal{C}'_m(I)$ and $E$ is the associated change of basis matrix, then $M' = EM$ and, for all $A$, $y'_A = \det(E) y_A$. Hence,

(1) The point $g_m([I])$ of $\mathbb{P}(W)$ defined by the collection of $y_A$ is independent of the choice of $\mathcal{C}_m(I)$.
(2) Whether or not any individual $y_A$ vanishes at $g_m([I])$ is likewise independent of this choice.
(3) We may always make this choice so that $M_{I,m}$ is in echelon form.

Example 2.2 For a monomial ideal, we may take the basis $\mathcal{C}_m$ to be monomial, too, and then the Hilbert matrix is particularly simple: it will have exactly one 1 in each row and be 0 otherwise. Thus, for a given $m$, there is exactly one nonzero Plücker coordinate, given by the Plücker set $A = \mathcal{C}_m$.

Example 2.3 Consider the ideal $I$ of two distinct points in $\mathbb{P}^2$. For instance $P = (1, 2, 3)$ and $Q = (5, 1, -4)$. Let $a, b, c$ be the coordinates on $\mathbb{P}^2$. Then we can view $I$ as $(c - 3a, b - 2a) * (a - 5b, c + 4b)$ and take

$$\mathcal{C}_2 = [2a^2 - 11ab + 5b^2, 8ab - 4b^2 + 2ac, 3a^2 - 15ab - ac + 5bc, 12ab + 3ac - 4bc - c^2] .$$

Ordering $\mathcal{B}_{S_2}$ as $[a^2, ab, ac, b^2, bc, c^2]$, we get:

$$M_{I,2} = \begin{pmatrix} 2 & -11 & 0 & 5 & 0 & 0 \\ 0 & 8 & 2 & -4 & -1 & 0 \\ 3 & -15 & -1 & 0 & 5 & 0 \\ 0 & 12 & 3 & 0 & -4 & -1 \end{pmatrix} \tag{2.4}$$

Then the Plücker point of $M_{I,2}$ is given by the following point, in which we have indexed the Plücker sets by the pair of monomials *omitted* to save space

| $\widehat{12}$ | $\widehat{13}$ | $\widehat{14}$ | $\widehat{15}$ | $\widehat{16}$ | $\widehat{23}$ | $\widehat{24}$ | $\widehat{25}$ | $\widehat{26}$ | $\widehat{34}$ | $\widehat{35}$ | $\widehat{36}$ | $\widehat{45}$ | $\widehat{46}$ | $\widehat{56}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | : -95 | : 99 | : -154 | : 209 | : 55 | : -18 | : 38 | : -13 | : -83 | : 108 | : -228 | : 22 | : 55 | : -132 |

Alternatively, the Plücker coordinates can be computed in `Macaulay2` as follows:

```
i1 : R = QQ[a..c];
i2 : I = intersect(ideal(c-3*a,b-2*a),ideal(a-5*b,c+4*b));
i3 : G = flatten entries super basis(2,I);
i4 : B = flatten entries basis (2,R);
i5 : M = matrix apply(G, i-> apply(B, j -> coefficient(B_j,G_i)))
o5 = | 11 -19 9  0   0   0  |
     | 0  1   0  -19 9   0  |
     | 0  0   11 0   -19 9  |
     | 0  0   0  12  -5  -2 |
```

The command `minors(4,M)` then gives the determinants of the $4 \times 4$ minors of $M$. However, the order in which `Macaulay2` lists the basis of $\bigwedge^4 S_2$ is not, in our experience, any obvious one. Note also that, in `Macaulay2`, we computed the intersection of the two ideals, rather than product. The result is not the ideal we computed by hand, but its saturation. The two Hilbert matrices are different but row-equivalent, and the two sets of Plücker coordinates agree up to a multiple of 121 in each coordinate (that we have supressed in our listing), hence represent the same point in $\mathbb{P}^{14}$.

## 3. Stability and state polytopes

**$T$-states and $T$-state polytopes.** We next want to focus on the action of $\mathrm{SL}(V) \cong \mathrm{SL}(N+1)$ on $W := \bigwedge^{P(m)} \mathrm{Sym}^m V$. The Hilbert–Mumford criterion says that $w \in W$ is $\mathrm{SL}(V)$ stable if and only if $w$ is $\lambda$-stable for every 1-parameter subgroup (henceforth 1-ps) $\lambda : \mathbb{G}_m \to \mathrm{SL}(V)$. If, in terms of a basis of $V$ with respect to which $\lambda$ diagonalizes as $\mathrm{diag}(t^{r_0}, \cdots, t^{r_N})$, the point $w$ has coordinates $(w_0, \ldots, w_N)$, we set

$$\mu^L(w, \lambda) := -\min\{r_i \mid i \text{ such that } w_i^* \neq 0\},$$

and $w$ is $\lambda$-stable if and only if $\mu^L(w, \lambda) < 0$.

Remark 3.1 A word about the minus in the definition of $\mu$. Our preferred sign convention for the index $\mu$ of a Hilbert point $w$ is that of [12], [2] and [22] in which we consider the Grassmanian as parameterizing $P(m)$-dimensional quotients of $S_m$, given by restriction to $H^0\big(X, \mathcal{O}_X(m)\big)$, and $w$ is stable if any $\lambda$ acts with negative weight on some non-zero coordinate of $w$. The minus sign has been inserted to compensate for the fact that here we will be calculating weights of the action of $\lambda$ on the degree $m$ piece of the ideal $I$ of $X$ which is dimension $\widehat{Q}(m)$. This, of course, gives rise to a quotient of dimension $\widehat{P}(m)$ and the complement of each Plücker set $A$ of monomials gives a basis of this quotient. But, because we are taking $m$ small, we can no longer identify the quotient with $H^0\big(X, \mathcal{O}_X(m)\big)$ except on the $l$-nice locus and it therefore seemed easier to us to simply work with $I_m$. This choice has no effect on the notion of $\mathrm{SL}(V)$-stability because the possibility of replacing $\lambda$ by its inverse means that $w$ is stable if and only if we can always find a non-zero coordinate of $w$ on which $\lambda$ acts with a weight of either sign.

The connection with Gröbner theory comes via another way of expressing the stability of $w$ with respect to the maximal torus $T$ of $\mathrm{SL}(V)$ determined by a choice of basis $B$ of $V$. Any character $\chi \in \mathrm{Hom}(T, \mathbb{G}_m)$ of $T$ may be written

$$\chi\big(\mathrm{diag}(d_0, \ldots, d_N)\big) = \prod_{i=0}^{N} d_i^{z_i}.$$

where the $z_i$ are integers, determined (since we are in $\mathrm{SL}(V)$) up to a common shift. Further, any representation $W$ of $T$ decomposes into a direct sum of character eigenspaces $W_\chi$, where $w \in W_\chi$ if and only if $t \cdot w = \chi(t) w$ for all $t \in T$.

Define the *$T$-state* $\mathrm{State}_T(w)$ of $w$ to be the set of characters for which the eigencomponent $w_\chi$ of $w$ is non-zero, and define the *$T$-state polytope* $\mathcal{P}_T(w)$ to be the convex hull of $\mathrm{State}_T(W)$ in $\mathrm{Hom}(T, \mathbb{G}_m)$.

The group of 1-parameter subgroups of $T$ is dual to its character group: $\lambda \cdot \chi$ is the $\lambda$-weight of $\chi$—the power of $t$ determined by the homomorphism $\chi \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$. Viewing $\lambda$ as giving a linear functional on $\mathrm{Hom}(T, \mathbb{G}_m)$, we may rephrase our discussion of the Numerical Criterion as saying that $w \in W$ is stable with respect to a 1-ps $\lambda$ in $T$ if and only $w$ has two non-zero eigencomponents $w_\chi$ whose character vectors lie on opposite sides of the hyperplane on which $\lambda$ vanishes. Thus we arrive at the following characterization of GIT stability:

**Criterion 3.2** *A vector $w \in W$ is $T$-stable iff the the trivial character lies in the interior of the state polytope and is $T$-strictly semistable iff the trivial character lies on the boundary of the state polytope.*

To interpret Criterion 3.2 for Hilbert points, first observe that each eigenspace $(S_m)_\chi$ of $S_m$ is spanned by a single $B$-monomial $M$ and, if we normalize the choice of the $z_i$ above by requiring that they sum to $m$, then we may identify the character $\chi$ and the exponent vector of $M$. The Plücker coordinates $y_A$ on $W$ likewise give an eigenbasis, although the eigenspaces are not necessarily 1-dimensional. If we now normalize so that the $z_i$ sum to $\widehat{Q}(m)m$, then we can identify the corresponding character $\chi_A$ with the sum of the exponent vectors of the $\widehat{Q}(m)$ monomials determined by $y_A$. For example, in $\bigwedge^2 \mathrm{Sym}^2 K[a, b, c, d]$, the wedge product $a^2 \wedge bc$ lies in the weight space corresponding to $W_{(2,1,1,0)}$.

Monomials and Plücker coordinates also diagonalize the actions of a 1-ps $\lambda$ of $T$ on $S_m$ and $W$. The weight $w_\lambda(M)$ of a monomial $M$ is the sum of the weights of its coordinate factors and the weight $w_\lambda(y_A)$ of a Plücker coordinate $y_A$ is the sum of the weights of the monomials in it. Moreover, these weights agree with the $\lambda$-weights of the corresponding characters.

Thus, we think of the characters as lying on the hyperplane

$$Z_m := \{z \in \mathbb{Z}^{N+1} \mid \sum_{i=0}^{N} z_i = mP(m)\}$$

This identifies the trivial character with the point in $\mathbb{Q}^{N+1}$ having all coordinates equal to $\frac{m\widehat{Q}(m)}{N+1}$. In the sequel, we will denote this point by $\mathbf{0}_m$ and call it the *barycenter* of $Z_m$.

To simplify two notations that we will use frequently, we write $\mathrm{State}_{T,m}(I)$ and $\mathcal{P}_{T,m}(I)$ for the $T$-state and the $T$-state polytope of $g_m([I])$, omitting the $T$ when possible.

**Criterion 3.3** *The $m^{\text{th}}$-Hilbert point $g_m([I])$ of an ideal $I$ is $T$-stable iff $\mathbf{0}_m$ lies in the interior of $\mathcal{P}_{T,m}(I)$ and is $T$-strictly semistable iff $\mathbf{0}_m$ lies on the boundary of $\mathcal{P}_{T,m}(I)$.*

Note that Criteria 3.2 and 3.3 only test $T$-stability. In Corollary 4.9, we will identify conditions under which we can extend this to $\mathrm{SL}(V)$-stability.

Example 3.4 The $m^{\text{th}}$-state of a monomial ideal $I$ is a single point, since there is only one nonzero Plücker coordinate. Unless this point equals $\mathbf{0}_m$, the $m^{\text{th}}$-Hilbert point of $I$ is unstable.

Example 3.5 If $X$ a hypersurface of degree $d$ in $\mathbb{P}^N$, we may take $d = m$—so $P(m) = 1$—and suppress the exterior power in $W$. Both the characters appearing in the decomposition of $W$ and its Plücker coordinates are then indexed by monomials $\prod_{i=0}^{N} x_i^{z_i}$ of degree $d$, and, viewed as lying in the plane $\sum_{i=0}^{N} z_I = d$, form the $d^{\text{th}}$ subdivision of an $N$-simplex.

Figure 3.6 shows this situation for a cuspidal plane cubic $X$ with equation $x^2z = y^3$. which is unstable with respect to the 1-ps $\rho$ shown. The set of characters appearing in the decomposition is are indicated by dots and the simplex that is their convex hull is the outlined triangle. The state polytope is the line segment joining the two monomials with non-zero coefficients in the equation.

These coordinates are both of which have weight 3 with respect to the 1-ps $\rho$ given by $\rho(t) = \mathrm{diag}(t^4, t, t^{-5})$ and hence this Hilbert point is unstable. The instability is reflected in the fact that $\mathcal{P}_{T,3}([X])$ does not contain $\mathbf{0}$.

A generic hypersurface in $\mathbb{P}^2$ would have a two-dimensional state polytope. The degeneracy of $\mathcal{P}_{T,3}([X])$ reflects the fact that this cuspidal cubic has an extra multigrading (it is toric, though not normal). But

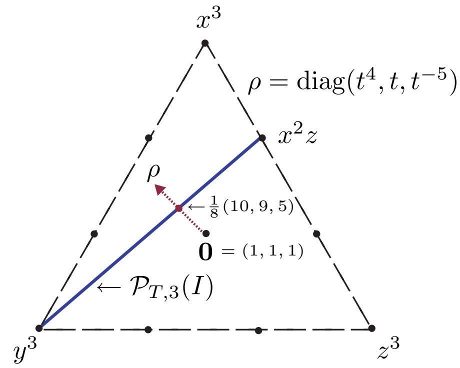


FIGURE 3.6: Degree 3 state polytope for a cuspidal plane cubic with ideal $I = \langle x^2z = y^3 \rangle$

adding an $x^3$ term to the equation, making the state polytope the upper sub-triangle subtended by $\mathcal{P}_{T,3}([X])$ would not affect the instability.

EXAMPLE 3.7 We continue with Example 2 of the previous section, the ideal of $[1:2:3] \cup [5:1:-4] \subset \mathbb{P}^2$. Every character with non-zero eigenspace contains a nonzero Plücker coordinates, and the state polytope is the two-dimensional hexagon pictured in Figure 3.8. Here the barycenter (indicated by the central solid circle) has coordinates $(\frac{8}{3}, \frac{8}{3}, \frac{8}{3})$.

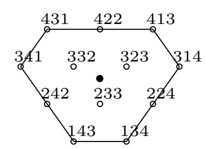


FIGURE 3.8: $\text{State}_2(I)$ for two general points in $\mathbb{P}^2$

**Vertices of state polytopes and initial ideals.** The number of Plücker coordinates grows quickly as the number of variables, the number of generators of the ideal, and $m$ grow. Thus it is impractical to compute the state polytope from definitions for all but the very simplest examples. The following results, modelled closely on analogous statements in [4], allow us to handle larger examples by giving a procedure for finding the vertices of $\mathcal{P}_{T,m}(I)$ that avoids the need to deal with interior Plücker coordinates.

Any 1-ps $\lambda$ in $T$ yields a non-strict order $\geq_\lambda$ on monomials:

$$M \geq_\lambda M' \iff w_\lambda(M) \geq w_\lambda(M').$$

Since the weights $w_i$ of $\lambda$ on $V$ are integers there will always be ties in large degree. But in any fixed degree $m$, $\geq_\lambda$ will give a total order for all $\lambda$ not lying on a finite collection of hyperplanes. If so, we say that $\lambda$ is $m$-generic. We will say that $\lambda$ is generic if it is $m$-generic for $\ell \leq m \leq m_P$.

**Lemma 3.9** *For any $I$ in $\widehat{\mathbb{H}}$ and any generic 1-ps $\lambda$, there is a unique Plücker set $A_\lambda$ of $\widehat{Q}(m)$ monomials such that:*

(1) *$y_{A_\lambda}$ is non-zero at $g_m([I])$.*
(2) *If $y_{A'}$ is any other Plücker coordinate non-zero at $g_m([I])$, then $w_\lambda(y_A) > w_\lambda(y_{A'})$.*

*Moreover, if $M_{I,m}$ is a $m$-Hilbert matrix for $I$ in echelon form, then the monomials in $A_\lambda$ span the $>_\lambda$-initial ideal $\text{in}_{>_\lambda}(I)$ of $I$ in degree $m$.*

*Proof.* This is the content of Lemma 3.3 and Corollary 3.4.(ii) of [4] and the proofs given there apply verbatim in our situation. □

**Definition 3.10** *For any generic 1-ps $\lambda$, we let $\chi_\lambda = \chi_{A_\lambda}$. In other words, $\chi_\lambda$ is the character given by summing the exponent vectors of the $\widehat{Q}(m)$ monomials in $\text{in}_{>_\lambda}(I)_m$. By (1) of Lemma 3.9, this character is an element of $\text{State}_m(I)$.*

**Theorem 3.11** *For any $m$-generic 1-ps $\lambda$, the character $\chi_\lambda$ is a vertex of the state polytope $\mathcal{P}_{T,m}(I)$. Conversely, if $\chi$ is any vertex of $\mathcal{P}_{T,m}(I)$, then the eigenspace $W_\chi$ is one dimensional and is spanned by the Plücker coordinate $y_{A_\lambda}$ for some $m$-generic $\lambda$. In particular, $\chi = \chi_\lambda$.*

*Proof.* The inequality in (2) of Lemma 3.9 shows that $\sum_{i=0}^N w_i z_i = w_\lambda(y_A)$ is a supporting hyperplane ($\chi_\lambda$ lies on it and all other $\chi'$ in $\mathrm{State}_m(I)$ lie on the negative side of it) and hence proves the first claim Conversely, any supporting hyperplane $\sum_{i=0}^N w_i z_i = b$ to $\chi$ may be perturbed so that the coefficients of its normal are the set of weights $w_i$ of a generic 1-ps $\lambda$. But then any Plücker coordinate $y_A$ lying in the $\chi$-eigenspace satisfies the conditions defining $y_{A_\lambda}$ in Lemma 3.9. The Lemma therefore implies that there is a unique such Plücker coordinate and that $\chi = \chi_\lambda$. The second claim follows. □

We note that, in general, the dimension of $W_\chi$ will be quite large. Already in Figure 3.8, the three interior characters have 2-dimensional eigenspace.

Theorem 3.11 is a weaker version of Theorem 3.1 of [4] which shows that if $m \geq m_P$, then the set for vertices of $\mathcal{P}_{T,m}(I)$ is canonically bijective to the set of initial ideals of $I$. For the small degrees that we are treating here where the map $g_m$ from an ideal to its degree $m$ graded piece is not injective, a surjection from initial ideals to vertices is all that we can hope for—and all we need for our applications.

Example 3.12 Let $X$ be the twisted cubic in $\mathbb{P}^3$ with ideal $I = \langle ac - b^2, ad - bc, bd - c^2\rangle$. Then $X$ has eight initial ideals:

(1) $\langle bd, ad, ac\rangle$
(2) $\langle c^2, ad, ac\rangle$
(3) $\langle c^2, bc, ac, a^2d\rangle$
(4) $\langle c^2, bc, b^3, ac\rangle$
(5) $\langle c^2, bc, b^2\rangle$
(6) $\langle bd, b^2, ad\rangle$
(7) $\langle bd, bc, b^2, ad^2\rangle$
(8) $\langle c^3, bd, bc, b^2\rangle$

$\mathrm{State}_2(I)$ has six vertices: initial ideals 3 and 4 agree in degree 2, as do initial ideals 7 and 8 initial ideals. For any $m \geq 3$, $\mathrm{State}_m(I)$ has eight vertices.

By [3, Proposition 1.8], given any multiplicative total order $>$ we can find a 1-ps $\lambda$ such that $>$ and $>_\lambda$ agree up to degree $m$. Hence,

**Corollary 3.13** *The state polytope $\mathcal{P}_{T,m}(I)$ is the convex hull of the set of $\chi_A$ as $A$ runs over all Plücker sets that are bases for the degree $m$ graded piece of the some initial ideal of $I$.*

Conveniently, Anders Jensen's program `gfan` [29] computes the set of initial ideals of $I$. Thus, if we compute the $m^{\text{th}}$ state of each initial ideal for sufficiently large $m$, we will have the state polytope. This is what the `Macaulay2` [30] package `StatePolytope` does.

We will not use the following geometric characterization of $A_\lambda$ but have found it helpful in thinking about the preceding results. The action of $\mathrm{SL}(V)$ on $V$ induces actions on the homogeneous polynomials of each degree on $V$ and hence on Hilbert scheme $\widehat{\mathbb{H}}$ and on the Grassmanian $\mathbf{Gr}_m$ for which the map $g_m$ is equivariant. Since $\widehat{\mathbb{H}}$ is projective, we can define an ideal $J$ giving a point of $\widehat{\mathbb{H}}$ by

$$[J] := \lim_{t\to 0} \lambda(t)\cdot [I]\,.$$

Lemma 3.9 says that $y_{A_\lambda}$ is the unique Plücker coordinate that is non-zero at $g_m([J])$ and hence that $g_m([J]) = g_M([\mathrm{in}_{>_\lambda}(I)])$. But all these arguments apply equally to any other degree between $\ell$ and $m_P$ so that, in all these degrees, $J$ and $\mathrm{in}_{>_\lambda}(I)$ are equal. Hence,

**Proposition 3.14** *$J = \mathrm{in}_{>_\lambda}(I)$ in degrees above $\ell$.*

## 4. Kempf's theory of the worst 1-ps

Let $w$ be a point of an $\mathrm{SL}(V)$ representation $W$. Already on page 64 in the first edition of [9], Mumford conjectured that if $w$ is unstable, there is a worst destabilizing 1-ps $\lambda$ as measured by the index $\mu(w,\lambda)$. In this section, we review the proof of this conjecture by Kempf [20] and Rousseau [23] but, to simplify, deal only with the linear situation we need in our applications. We follow the treatment of Kempf which contains some complementary results that allow us to reduce the $\mathrm{SL}(V)$-stability of points $w \in W$ with suitably large stabilizer $\mathrm{Stab}(w)$ to their $T$-stability for a special torus determined by this stabilizer.

We begin by reviewing some of the background of Kempf's arguments. First, some easy covariance properties.

**Lemma 4.1** *For any $g \in \mathrm{SL}(V)$,*

(1) $\mathrm{State}_T(g\cdot w) = g\,\mathrm{State}_T(w)g^{-1}$.
(2) $\mu(w,\lambda) = \mu(g\cdot w, g\cdot\lambda\cdot g^{-1})$.

*Proof.* The first statement is Lemma 3.2.d) of [20] and the second follows immediately from it. □

Next, let $F_\lambda$ be the $\lambda$ weight filtration on $V$ and $P_\lambda$ be the parabolic subgroup of block upper triangular matrices in $\mathrm{SL}(V)$ preserving $F_\lambda$. Equivalently, $P_\lambda$ consists of those $p \in \mathrm{SL}(V)$ for which the limit $\lim_{t\to 0}\lambda(t)\cdot p\cdot\lambda^{-1}(t)$ exists.

**Lemma 4.2** *If $g \in \mathrm{SL}(V)$ and $p \in P_\lambda$ then,*

(1) $P(g\lambda g^{-1}) = gP_\lambda g^{-1}$.
(2) $p \in P_\lambda \iff P(p\lambda p^{-1}) = P_\lambda$.
(3) *If $p \in P_\lambda$, $\mu(w,\lambda) = \mu(w, p\lambda p^{-1}) = \mu(p^{-1}\cdot w,\lambda)$. Hence, $\mu(w,\lambda) = \mu(p\cdot w,\lambda)$.*

*Proof.* The first statement follows directly from the characterization of $P_\lambda$ in terms of limits and then the second follows because any parabolic subgroup is its own normalizer.

The last assertion is trickier. Our proof follows that of Lemma 3.3.e) from [20]. The final equality follows from the first two by inverting $p$. By Lemma 4.1(2), the second equality follows from the first. For this, the key point is the following claim: if, as $t \to 0$, $\lambda(t)p^{-1}\lambda^{-1}(t) \to p_0^{-1}$ and $t^{-r}\lambda(t)\cdot w \to w_0$, then $t^{-r}p\lambda(t)p^{-1}\cdot w \to pp_0^{-1}\cdot w_0$. This follows because

$$t^{-r}p\lambda(t)p^{-1}\cdot w = p\left(\lambda(t)p^{-1}\lambda^{-1}(t)\right)\left(t^{-r}\lambda(t)\cdot w\right) \to p\;p_0^{-1}\cdot w_0$$

Since $\mu(w,\lambda)$ is the largest $r$ such that $\lim_{t\to 0} t^{-r}\lambda(t)\cdot w$ exists, the claim shows that if $p \in P_\lambda$, then $\mu(w,\lambda) \le \mu(w,p\lambda p^{-1})$ and, by symmetry, 3 follows. □

Replacing $\lambda(t)$ by $\lambda_k(t) := \lambda(t^k)$ for any positive integral $k$ scales all weights by $k$ without affecting their signs. This stability with respect $\lambda$ and $\lambda_k$ are equivalent but $\mu(w,\lambda_k) = k\mu(w,\lambda)$. To have a measure of badness that equates $\lambda(t)$ and $\lambda_k(t) := \lambda(t^k)$, we need to normalize the Hilbert-Mumford index $\mu$. To do this, we simply choose a conjugation-invariant norm $||\cdot||$ on one parameter subgroups—for $\mathrm{SL}(V)$, we can take $||\lambda|| := \left(\sum_{i=0}^N w_i^2\right)^{\frac{1}{2}} r$ and define $\widehat{\mu}(w,\lambda) := \frac{\mu(w,\lambda)}{||\lambda||}$. We also define $\overline{\mu}(w) := \sup_\lambda \widehat{\mu}(w,\lambda)$. A priori, it is not clear either that $\overline{\mu}(w)$ is finite or, if it is, that this sup is achieved. A *worst* $\lambda$ for $w$ is one for which the function $\widehat{\mu}(w,\lambda)$ achieves this maximum value.

In the when case $X$ is a representation $W$, Theorem 3.4 of [20] says that:

**Theorem 4.3** *If $w$ is an unstable point of $W$, then*

(1) *There is an indivisible 1-ps $\lambda$ such that, if $\lambda'$ any other 1-ps, then $\widehat{\mu}(w,\lambda) \ge \widehat{\mu}(w,\lambda')$. Hence $\overline{\mu}(w)$ is finite and equal to $\widehat{\mu}(w,\lambda)$.*
(2) *The indivisible $\lambda'$ for which $\widehat{\mu}(w,\lambda') = \widehat{\mu}(w,\lambda)$ are exactly those for which $\lambda' = p^{-1}\lambda_w p$ for some $p \in P_\lambda$. In particular, $P_{\lambda'} = P_\lambda$ and we can write $P_w$ for $P_\lambda$*

(3) *The set of all $\lambda'$ as in (2) is a principal homogeneous space under the unipotent radical of $P_w$ and every maximal torus $T$ of $P_w$ contains a unique such $\lambda'$.*

In view of the fact Lemma 4.1(2) and Lemma 4.2(3), we can informally summarize this result as saying that worst one-parameter subgroups exist and are as unique as possible. The complementary result we need is:

**Proposition 4.4** ([20, Corollary 3.5]) *Let $w \in W$ be an unstable point with associated parabolic subgroup $P_w$. Then $P_w$ contains $\mathrm{Stab}_w(\mathrm{SL}(V))$.*

*Proof.* For any $g \in \mathrm{SL}(V)$, $g \cdot w$ is also unstable and hence determines a parabolic subgroup $P_{g\cdot w}$. The Proposition will follow, if we show that $gP_wg^{-1} = P_{g\cdot w}$, because any parabolic subgroup is its own normalizer.

By Lemma 4.1(2) and the conjugation invariance of the norm $||\cdot||$, the (indivisible) worst one-parameter subgroups for $g \cdot w$ are exactly the $g$-conjugates of those for $w$. Let $\lambda$ be one of the latter. This gives the middle equality in $P_{g\cdot w} = P_{g\lambda g^{-1}} = gP_\lambda g^{-1} = gP_w g^{-1}$ and the first and last equalities follow from Lemma 4.2(1). □

Kempf applies these results to conclude stability of Chow and Hilbert points of abelian varieties and homogeneous spaces ([20, Cor. 5.2 and 5.3)]: the representations of the automorphism groups of these varieties are irreducible, so the stabilizer is not contained in any nontrivial parabolic, so these must be GIT stable.

There are very few examples of pluricanonically embedded smooth curves with an automorphism group acting via an irreducible representation. For instance, a full list of canonical curves with this property is found in [6, App. B]; the highest genus example is $g = 14$. Examples are even rarer as $\nu$ increases. So, Kempf's strategy must be modified if it is to be applied to moduli of curves.

Here is how we weaken the irreducibility hypothesis.

**Definition 4.5** *Fix $w \in W$. We say that $w$ is* multiplicity free *with respect to a finite subgroup $G$ of $\mathrm{Stab}_{\mathrm{SL}(V)}(w)$ if, in the representation of $G$ on $V$, no $G$-irreducible $R$ has multiplicity greater than 1. When, as in our applications here, $G = \mathrm{Stab}_{\mathrm{SL}(V)}(w)$ we will simply say that $w$ is* multiplicity free.

The key consequence of this property is that $V$—indeed, any $G$-invariant subspace $U$ of $V$—has a *canonical* decomposition as a direct sum of $G$-irreducible subrepresentations of $V$. Such a decomposition, of course, exists, for any finite $G$ by complete reducibility. But when $w$ is multiplicity free, there is, for each $R$ appearing in $V$, a canonical subrepresentation $U_R$ isomorphic to $R$. Every $U$ is then the direct sum of those $U_R$ for $R$ occurs in $U$.

**Definition 4.6** *We say that a basis $B$ of $V$ or the associated torus $T = T_B$* determines stability *for $w$ if:*

(1) *There is a subgroup $G \subset \mathrm{Stab}_{\mathrm{SL}(V)}(w)$ such that $w$ is multiplicity free with respect to $G$.*
(2) *The basis $B$ is the (disjoint) union of bases $B_R$ for each of the $G$-irreducible representations $U_R$ occurring in $V$.*

The justification for this terminology is:

**Proposition 4.7** *If $T$ determines stability for $w$ and $w$ is $T$-semistable, then $w$ is $\mathrm{SL}(V)$-semistable.*

*Proof.* We prove the contrapositive—if $w$ is $\mathrm{SL}(V)$-unstable, then $w$ is $T$-unstable—by showing that then $T$ is a torus of $P_w$.

So suppose that $w$ is multiplicity free and unstable. Then Proposition 4.4 says that $G$ lies in $P_w$ and hence fixes the associated filtration $F$. We can thus write $F$ as a strictly nested sequence $V = U_0 \supset U_1 \supset \cdots \supset U_h \subset \{0\}$ of $G$-invariant subspaces of $V$. Each of these is a direct sum of a subset of the $G$-irreducibles occurring in $V$. Therefore, the basis $B$ is compatible with the filtration $F$ and, in turn $T$ is a torus of $P_w$. □

We now apply this to Hilbert points. Let $X$ be an $l$-nice subscheme of $\mathbb{P}(V)$ with ideal $I$ and let $\mathrm{Aut}_V(X) \subset \mathrm{Aut}(X)$ be the subgroup of consisting of elements that act linearly on $V$ fixing $X$. Suppose that $\mathrm{Aut}_V(X)$ is finite and the representation of $\mathrm{Aut}_V(X)$ on $V$ is multiplicity free—in our applications $\mathrm{Aut}_V(X) = \mathrm{Aut}(X)$. For any $m \geq l$, the group $\mathrm{Aut}_V(X)$ lies in the $\mathrm{SL}(V)$-stabilizer of $g_m([I])$ so the pair $\big(g_m([I]), \mathrm{Aut}_V(X)\big)$ is multiplicity free in the sense of Definition 4.5, independently of $m$.

**Definition 4.8** *Under these hypotheses of the preceding paragraph, we say that $X$ is* multiplicity free *and that any torus $T$ constructed as in Proposition 4.7* determines stability for *$X$.*

Combining the Proposition with Criterion 3.3 gives:

**Corollary 4.9** *If $T$ determines stability for $X$, then the $m^{\mathrm{th}}$-Hilbert point $g_m([I])$ of $X$ is $\mathrm{SL}(V)$-stable* [resp: $\mathrm{SL}(V)$*-strictly semistable*] *if and only if the barycenter $\mathbf{0}_m$ lies in the interior* [resp: *the boundary*] *of the state polytope $\mathcal{P}_{T,m}(I)$.*

In an early phase of our research, we thought that we needed multiplicity-free representations of $\mathrm{Aut}(X)$ on $W$, rather than $V$. We communicated this misconception in a number of conversations and here apologize to those whom we misled.

Remark 4.10 We should emphasize that multiplicity free Hilbert points are extremely special—in general, such points have trivial stabilizer. For example, a general smooth curve of genus $g$ has a trivial automorphism group, so for any embedding, the corresponding representation is $N+1$ copies of the trivial representation of the trivial group. So this strategy can only prove directly the stability of special curves. On the other hand, by the openness of GIT stability and the coarseness of the Zariski topology, proving that a single smoothable subscheme in any component of the Hilbert scheme is stable proves that a general smooth subscheme on that component is stable! As Gieseker's construction of $\overline{M}_g$, and many others modeled on it (cf. [22]), show, such a statement is often enough, when there is a main component containing smooth equidimensional subschemes to allow the construction of a GIT quotient to be completed by indirect arguments.

For the rest of the paper we specialize to the case where $X$ is a curve, though many arguments will continue to apply more generally. To make this switch clear we write $C$ for $X$, continuing to denote its ideal by $I$. In looking for examples in this case, the next step is therefore clear. Find special models $C \subset \mathbb{P}(V)$ of curves that are multiplicity free and decide when their $m^{\mathrm{th}}$-Hilbert points are stable, at least for small $m$, by computing $\mathcal{P}_{T,m}(I)$ for some $T$ that determines stability. A natural set of models to consider are pluricanonical ones, since for these $\mathrm{Aut}_V(C) = \mathrm{Aut}(C)$. We have focussed here on bicanonical models for three reasons.

**Definition 4.11** *We say that a nodal curve $C$ is $\nu$-*multiplicity free *if its $\nu$-canonical model is multiplicity free. We will mainly be interested in the case $\nu = 2$ when we say that $C$ is* bicanonically multiplicity free*.*

Our applications involve only bicanonical models for three reasons. First, as mentioned in the introduction, the conjectural next stages in the log minimal model program of Hassett and Hyeon [16, 17] depend on understanding the stability of bicanonical Hilbert points of degree at or below 6. Second, bicanonical embedding dimensions are small enough that it is practical to compute state polytopes in these degrees, as least for curves of small genus $g$.

Finally, MacLachlan [21, Theorem 4] shows that, if a curve of genus $g$ has an *abelian* automorphism group, its order can be at most $4g+4$. Hence no such curve can be $\nu$-multiplicity free for any $\nu > 2$ unless is has very small genus; all the irreducibles have dimension 1 and the $\nu$-canonical series has dimension $(2\nu - 1)(g - 1)$. Multiplicity free curves with non-abelian automorphism groups seem likewise to be extremely rare. [Dave I still think it worth your while to write up a list as an appendix (under your name only). If so, point to it here. IM] The sum of the squares of the dimensions of the irreducibles equals the order of the group so the sum of the dimensions themselves is typically much less, and this more than compensates for the largely theoretical extra headroom given by Hurwitz' bound of $84(g-1)$ for their orders. In fact, our examples, reviewed in Section 6, are a family $\mathcal{W}_g$ of hyperelliptic curves, one in each genus $g$, called Wiman curves whose automorphism groups are cyclic of order—surprisingly, in view of Maclachlan's bound—$4g+2$.

## 5. Complementary approaches to checking stability

We are almost ready to describe our applications of the preceding results to check, by symbolic calculations, GIT stability of certain bicanonically multiplicity free nodal curves $C$ of small genus with respect to small $m$ linearizations.

Given such an $C$ with ideal $I$, what we'd like to do is to compute the state polytope $\mathcal{P}_{T,m}(I)$ with respect to a distinguished torus of Corollary 4.9 and check whether the barycenter $\mathbf{0}_m$ lies in its interior, boundary or exterior. But in practice, already, in some cases with $g=4$, this plan is impractical to carry out: there are too many initial ideals and we are unable to completely compute $\mathcal{P}_{T,m}(I)$.

In this section, we explain some additional tricks that we use to settle such cases in some of our examples.

**A Monte Carlo Pseudo-Algorithm.** We can often, however, check GIT stability without computing the entire state polytope. If we can compute *any* set $\Xi$ of points such that $\Xi \subset \mathrm{State}_m(I)$ and the convex hull $\overline{\overline{\Xi}}$ contains $\mathbf{0}_m$, then we know $C$ is $m$-Hilbert semistable (even stable if $\mathbf{0}_m$ lies in the interior of $\overline{\overline{\Xi}}$). We say such a $\Xi$ checks semistability of $X$.

**Pseudo-Algorithm 5.1** *To verify that $C$ is $m$-Hilbert semistable, compute pseudo-random weight vectors $\lambda$ and then find the associated degree $m$ initial ideals* $\mathrm{in}_{>_\lambda}(I)_m$, *let $\Xi$ be the set of characters $\chi_\lambda$ in* $\mathrm{State}_M(I)$ *these determine. If $\overline{\overline{\Xi}}$ contains $\mathbf{0}_m$, stop.*

This is, of course, only a pseudoalgorithm because it will never terminate if $C$ is actually $m$-Hilbert *un*stable. In fact, even if $C$ is $m$-Hilbert semistable, we cannot be sure it will terminate: we may simply not have generated vectors in enough directions to produce a $\Xi$ that checks semistability. In practice, however, we have not encountered either problem. Guided by the predictions of the log minimal model program, we have been able to apply it only to testing $C$ that actually were $m$-Hilbert semistable. And, in all the examples we have run, Pseudo-Algorithm 5.1 has produced a $\Xi$ that checks stability quickly, typically in a tiny fraction of the time required for our calculation of the full state polytope to complete—or fail to complete due to hardware limitations.

In fact, we have also been able to settle $m$-Hilbert stability in all cases where $C$ turned out to be unstable. This is made possible by a fundamental asymmetry of stability calculations. Proving semistability is hard; proving instability is easy because it can be done by "intelligent guesswork".

If we can guess a destabilizing 1-ps $\lambda$, it is strightforward to prove instability. To check such a guess, it suffices to compute an the $\mathrm{in}_{>_\lambda}$ initial ideal (adding a tie-breaking procedure if necessary). The ideal $I$ is $m$-Hilbert unstable if and only if the monomial basis of this ideal in degree $m$ has negative weight: by Lemma 3.9, this weight equals $\mu(g_m(I),\lambda)$. The `Macaulay2` function `MUm` computes $\mu$ by this method.

In the same spirit, the log minimal model program gives a geometric description of those curves whose instability it predicts. This geometry suggests both the filtration and the weights of a candidate destabilizing 1-ps $\lambda$ and these candidates have proven to be destabilizing in all our examples. In practice, it is often possible to check instability with respect to these $\lambda$ deductively because $\lambda$-weights have a geometric interpretation. This pattern is familiar to those who have computational experience, symbolic or deductive, with Hilbert stability.

**The Parabola Trick.** For a fixed curve $C \in \mathbb{P}(V)$, the complexity of computing the state polytope $\mathcal{P}_{T,m}(I)$ grows quite rapidly with $m$ because $\binom{m+N}{m}$, the number of monomials of degree $m$, grows like $N^m$ for $N \gg m$. This often means that we can check $\ell$-Hilbert stability for some $\ell$ but not $m$-Hilbert stabiolity for $m$ a somewhat larger, but still small, degree of greater geometric interest. Typically $\ell = 2$ and, with applications to the log minimal model program for $\overline{M}_g$ in mind, $m \leq 6$. The following Proposition sometimes lets us deduce what we want to know from what we can compute. It introduces the somewhat technical notion of $h$-stability which arises in [16, Definition 2.6]. We will not need to enter into the details here, except to remark that all smooth curves are $h$-stable as are "most" Deligne-Mumford stable nodal curves.

**Proposition 5.2** (The Parabola Trick) *If $C$ is an $\ell$-nice bicanonical curve which is also both $h$-stable and $\ell$-Hilbert stable, then $C$ is $m$-Hilbert stable for any $m \geq \ell$.*

*Proof.* All the work has been done in [16]. We first use the fact that, if $C$ is $\ell$-nice, then for any non-trivial 1-ps $\lambda$, the weight function $w_\lambda(m)$ is computed by a quadratic polynomial for $m \geq \ell$ and that this polynomial has the form $w_\lambda(m) = a(m-1)(m-r)$ for some rational $a$ and $r$. This is proved for $\ell = 2$ in Proposition 3.17, but the same proof works, mutatis mutandi, for any $\ell \geq 2$. Since, by Theorem 2.14, any $h$-stable curve $C$ is asymtotically stable (i.e. stable for $m \gg 0$), we know that $a > 0$. But then $w_\lambda(\ell) > 0$ if and only if $r \leq \ell$, and if this is true, then $w_\lambda(m) > 0$ for any $m \geq \ell$. Since $\lambda$ is arbitrary, $C$ is $m$-Hilbert stable. □

## 6. Background on Wiman Curves

Looking ahead to the next section, our main source of computational examples will be a sequence $\mathcal{W}_g$ of hyperelliptic curves called Wiman curves. In this section, we develop the theoretical background on these curves that we will need in these applications. To begin with, we need to have suitable equations for their pluricanonical models and this part of the story depends only on their being hyperelliptic.

**Pluricanonical equations of hyperelliptic curves.** So let $C$ be a smooth hyperelliptic curve. Let $\phi_\nu : C \to \mathbb{P}^{V_\nu} = \mathbb{P}^{N_\nu}$ be the $\nu$-canonical embedding in $V_\nu = H^0\big(C, K_C^{\otimes\nu}\big)^\vee$. By Riemann-Roch, $N_\nu + 1 = (2\nu-1)(g-1)$ and $\phi_\nu(C)$ has degree $d = \nu(2g-2)$. What are equations for $\phi_{\nu K}(C)$? It is easy to write down such equations—indeed, equations for the embeddings of hyperelliptic curves by more general linear systems. Here we do so in a form convenient for our applications in the next section, following Stevens [24, pp. 137–138] and Eisenbud [8]. To simplify notation, we fix $\nu$ and omit it where possible.

Let $\pi : C \to \mathbb{P}^1$ be the $g^1_2$ on $C$. For convenience, write $k := \nu(g-1)$ and $e := g+1$. Then $\phi(C)$ lies on the scroll $S = \mathbb{P}_{\mathbb{P}^1}(\pi_*(K_C^{\otimes\nu}))$, where $\pi_*(K_C^{\otimes\nu}) \cong \mathcal{O}(k) \oplus \mathcal{O}(k+e)$; thus $S \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$.

Let $C$ be given by the affine equation $y^2 = f(x)$, where $f(x)$ is polynomial in $x$ of degree $2g+2$ (or $2g+1$, if the point infinity is a branch point). Then a basis of $H^0(C, K_C^{\otimes\nu})$ is given by:

$$B_\nu := \{1, x, x^2, \ldots, x^k, y, yx, yx^2, \ldots, yx^{k-e}\}\,. \tag{6.1}$$

We check: $(k+1)+(k-e+1) = 2\nu(g-1)-(g+1)+2 = (2\nu-1)(g-1) = h^0(C, K_C^{\otimes\nu})$. Abusing notation, we use these basis elements as variables on $\mathbb{P}^N$.

Equations for the scroll are also classical; modern references are [15, Exercise 9.11] and [1, pp. 96–100]. Suppose $(\nu-1)(g-1) > 2$, so that $k > e$. Then the scroll equations are given by the $2 \times 2$-minors of the *deleted catalecticant matrix*

$$M := \left(\begin{array}{cccc|cccc} yx^{k-e} & yx^{k-e-1} & \cdots & yx & x^k & x^{k-1} & \cdots & x \\ yx^{k-e-1} & yx^{k-e-2} & \cdots & y & x^{k-1} & x^{k-2} & \cdots & 1 \end{array}\right). \tag{6.2}$$

Write $I_S$ for the ideal generated by the $2 \times 2$-minors of $M$.

Next, choose a set $Q$ of quadrics encoding the following equations:

$$\begin{array}{ccc} y^2 & = & f(x) \\ y^2x & = & xf(x) \\ & \vdots & \\ y^2x^{2(k-e)} & = & x^{2(k-e)}f(x). \end{array} \tag{6.3}$$

The particular choices of quadrics used to encode these equations won't matter once these are combined with the scroll equations. Write $I_Q$ for the ideal generated by these quadrics. The references cited above yield, in particular:

**Lemma 6.4** *Suppose* $\deg f(x) = 2g+1$*, i.e. the hyperelliptic cover yielding* $C$ *is branched at* $\infty$*. Then* $I(\phi_\nu(C)) = I_S + I_Q$*. That is, the ideal of* $\phi_\nu(C)$ *is given by the scroll equations (the* $2\times 2$*-minors of* $M$*) together with* $2(k-e)+1$ *additional equations coming from* $Q$*.*

**Corollary 6.5** *The equations of* $\phi_\nu(C)$ *above are compatible in the sense that for any* $\nu' \geq \nu \geq 2$*, the projection from* $\mathbb{P}^{V_{\nu'}}$ *to* $\mathbb{P}^{V_\nu}$ *by "forgetting the extra variables" maps* $\phi_{\nu'}(C)$ *to* $\phi_\nu(C)$*.*

EXAMPLE 6.6 To illustrate, Lemma 6.4, let's find the bicanonical equations $\mathcal{W}_4$ which, as we'll see in (6.9) is the curve given by $y^2 = x^9 - 1$.

Here $k = 6$, $e = 5$, and $N = 8$. We coordinatize $\mathbb{P}^8$ as follows:

$$\begin{array}{ccccccccccccccccc} 1 & : & x & : & x^2 & : & x^3 & : & x^4 & : & x^5 & : & x^6 & : & yx & : & yx^2 \\ a & : & b & : & c & : & d & : & e & : & f & : & g & : & h & : & i \end{array}$$

The deleted catalecticant matrix is

$$M := \begin{pmatrix} yx & | & x^6 & x^5 & x^4 & x^3 & x^2 & x \\ y & | & x^5 & x^4 & x^3 & x^2 & x & 1 \end{pmatrix} = \begin{pmatrix} i & | & g & f & e & d & c & b \\ h & | & f & e & d & c & b & a \end{pmatrix},$$

yielding[1]

$$\begin{aligned} I_S = (&-gh+fi, -fh+ei, -f^2+eg, -eh+di, -ef+dg, -e^2+df, -dh+ci, \\ &-df+cg, -de+cf, -d^2+ce, -ch+bi, -cf+bg, -ce+bf, -cd+be, -c^2+bd, \\ &-bh+ai, -bf+ag, -be+af, -bd+ae, -bc+ad, -b^2+ac) \end{aligned}$$

We may encode the equation $y^2 = x^9 - 1$ as $h^2 - dg + a^2$, the equation $y^2x = x^{10} - x$ as $hi - eg + ab$, and the equation $y^2x^2 = x^{11} - x^2$ as $i^2 - fg + ac$. Then the ideal we seek is

$$\begin{aligned} (&-gh+fi, -fh+ei, -f^2+eg, -eh+di, -ef+dg, -e^2+df, -dh+ci, -df+cg, -de+cf, \\ &-d^2+ce, -ch+bi, -cf+bg, -ce+bf, -cd+be, -c^2+bd, -bh+ai, -bf+ag, -be+af, \\ &-bd+ae, -bc+ad, -b^2+ac, h^2-dg+a^2, hi-eg+ab, i^2-fg+ac). \end{aligned} \tag{6.7}$$

[In working with these ideals, one notices the following phenomenon.

**Claim 6.8** *Order the elements in* $Q$ *in any order.* $V(I_S + (q_0))$ *has dimension 1. Each time an additional quadric* $q_i$ *is added, the degree and the arithmetic genus decrease by 1, until one reaches a smooth curve of degree* $4g-4$ *and genus* $g$*.*

Clearly, $V(I_S + (q_0))$ must contain the hyperelliptic curve plus a number of secant lines, and as we add more generators to the ideal, we are gradually getting rid of the secant lines. But we have not proven this carefully yet, so we leave it as a claim. (This may be partially or fully explained in [8]; we have not read it carefully enough yet.) DS]

**Bicanonical multiplicity-freeness of Wiman curves.** We write $W_g$ for the *Wiman curve of type I in genus* $g$. These curves are named for Anders Wiman who, in 1895 in his first published paper[2] [26], showed that $W_g$ has the cyclic automorphism group of largest order $4g+2$ amongst all smooth curves of genus $g$.

The curve $W_g$ is the smooth hyperelliptic curve given by the affine equation

$$y^2 = x^{2g+1} - 1\,. \tag{6.9}$$

[1]In `Macaulay2`, this i easily done by first entering $M$ using `matrix {{i,g,f,e,d,c,b},{h,f,e,d,c,b,a}}` and then obtaining $I_S$ by invoking `minors(2,M)`.

[2]His last appeared 59 years later!

It is often convenient to think of $W_g$ as the hypersurface in $\mathbb{P}(1, g+1, 1)$ given by $y^2 = x^{2g+1}z - z^{2g+2}$ and, when we do, we call $[1:0:0]$ the branch point at infinity. As we have already remarked, $\mathrm{Aut}(W_g)$ is cyclic of order $4g+2$: fixing a primitive $(4g+2)^{\mathrm{nd}}$ root of unity $\zeta$ determines a generator $\sigma \in \mathrm{Aut}(W_g)$ that acts with weight $2g+1$ on $y$ (that is, as $-1$) and with weight 2 on $x$. We key facts we will need about this action are summarized in:

**Proposition 6.10** *Let $B_2$ be the basis of $V = H^0(C, K^{\otimes 2})$ given by (6.1) and let $T$ be the corresponding torus in* $\mathrm{SL}(V)$.

1. $\mathrm{Aut}(W_g)$ *fixes the branch point at infinity.*
2. *The elements of $B_2$ are eigenvectors for the action of $\sigma$ on $H^0(W_g, K^{\otimes 2})$ with distinct powers of $\zeta$ as eigenvalues.*

*Hence, the bicanonical model of each curve $W_g$ is multiplicity free and the torus $T$ determines stability for it.*

*Proof.* Since $\sigma$ clearly fixes the point at infinity, the first statement is clear from the description of its action above. This description also shows that its action on $H^0(C, K^{\otimes 2})$ in the basis $B_2$ of (6.1) is by $\zeta^{2i}$ on $x^i$ for $i = 0, 1, \ldots, k = 2g-2$ and by $\zeta^{2i+2g+1}$ on $yx^i$ for $i = 0, 1, \ldots, k-e = g-3$. The gives the second statement and it, in turn, shows that the elements of $B_2$ span invariant lines on each of which $\mathrm{Aut}(W_g)$ acts by a different character. From this, the final claims follow immediately. □

## 7. Results

We prove that the bicanonical genus 3 Wiman curve $W_3$ (defined below) is unstable for $m = 2$ and stable for $m \geq 3$ (which matches the predictions based on [19]). We prove that the genus 4 and 5 Wiman curves $W_4$ and $W_5$ are stable for all $m \geq 2$ (which matches predictions of Hassett and Hyeon). We also prove that a specific genus 5 nodal curve with a genus two tail is unstable for $m < 6$, semistable for $m = 6$, and stable for $m \geq 7$.

### Stability of some smooth Wiman curves.

**Example: the genus 3 Wiman curve.** Genus 3 bicanonical curves are not explicitly covered by Section 6, since $(\nu-1)(g-1) = 2$, or equivalently, $k = e$. But we can stretch the algorithm there to cover this case, too: instead of the curve lying on a scroll given by a deleted catalecticant matrix, in genus 3, the curve lies on a cone over the rational normal curve given by a catalecticant matrix. To this we can add a quadric encoding $y^2 = f(x)$, yielding: the ideal for $W_3$ in $K[a,b,c,d,e,f]$ is $(ac-b^2, ad-bc, ae-bd, bd-c^2, be-cd, ce-d^2, f^2-ab+e^2)$.

In this case it is actually possible to compute the entire state polytope, so it is not necessary to use Monte Carlo methods. `gfan` [29] finds 4615 initial ideals; interestingly, while $I$ is generated by quadrics, some of the initial ideals have much higher regularity—one of the initial ideals has a generator of degree 19. (The Gotzmann number for the Hilbert polynomial $8t-2$ is 26.) We find that for $m = 2$, $W_3$ is unstable, and for integers $3 \leq m \leq 19$, $W_3$ is stable.

This corroborates predictions based on work by Hyeon and Lee ([19, Proposition 19]. They find that for $g = 3$, divisors of slope $\leq 28/3$ contract the hyperelliptic locus. On the other hand, we can compute the polarization on the quotient: it is

$$[6\nu^2 m - 2\nu m - 2\nu + 1]\lambda - [\frac{1}{2}\nu^2 m]\delta$$

which for $\nu = 2$ gives $[20m-3]\lambda - [2m]\delta$, or slope $\frac{20m-3}{2m}$. We solve $\frac{20m-3}{2m} = \frac{28}{3}$ to obtain $m = 9/4$. Thus, one predicts that a hyperelliptic curve is unstable for $m = 2$ and stable for $m \geq 3$, which matches what is found.

We can go one step further in this case: for $m = 2$ we can find the closest point (or *proximum*) on the state polytope to the barycenter. This can be computed using the Maple package `Convex` ([27]). The result is $\vec{p} = (\frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{10}{5})$. The average of these entries is $7/3$ (the barycenter), and $\vec{p} - (7/3, \ldots, 7/3) = (1/15, \ldots, 1/15, -1/3)$. Thus, the worst 1-ps is one which scales the span of the rational normal curve with equal weights and scales the cone with complementary weight.

We check: using the `MUm` function from [18], and $w = (10, 10, 10, 10, 10, 12)$ we get $\mathtt{MUm}(I, w, 2) = -4$ and $\mathtt{MUm}(I, w, 3) = 24$. By [16] Prop 3.17, this gives $\mu([W_3]_m, \lambda) = 4(m-1)(4m-9)$. Thus this 1-ps is destabilizing for $m < 9/4$.

What about $m > 19$? Well, by results of Gieseker we know that smooth curves of degree $d > 2g + 1$ embedded by complete linear systems are asymptotically GIT stable. Moreover, for a given 1-ps $\lambda$, we know that the weight of $\lambda$ with respect to the linearization given by $m$ is a quadratic polynomial in $m$, and one of its roots is $m = 1$ ([16] Prop. 3.17). Gieseker's theorem tells us the leading term has positive sign. Thus, if our polynomial is positive for some $m_0 > 1$, we know it is positive for all $m \geq m_0$. Thus, for hyperelliptic curves, we can use $m_0 = 3$.

Is this the best we can do? What happens for $9/4 < m < 3$? We predict that the worst 1-ps for $m = 2$ is the last one to become stable as $m$ increases, and hence $W_3$ is stable for $m > 9/4$, but we don't have a clear enough picture for fractional linearizations to prove this yet.

**Example: the genus 4 Wiman curve.** We computed the ideal of $W_4$ in (6.7).

Using the Monte Carlo strategy in `Macaulay2` [30], we find that $W_4$ is stable for $m = 2, 3, 4, 5, 6, 7$. As discussed above, these calculations, combined with the arguments of [12, 16] imply that $W_4$ is stable for all $m \geq 2$.

For $g \geq 4$ Hyeon predicts (private communication) that divisors of slope $\leq 9$ contract the hyperelliptic locus. Solving $\frac{20m-3}{2m} = 9$ yields $m = 3/2$, so the prediction is that this curve should be stable for all $m \geq 2$, which matches what is found.

**Example: the genus 5 Wiman curve.** $W_5$ behaves just like for $W_4$. The ideal is obtained using the methods of Section 6. We find that $W_5$ is stable for $m = 2, 3, 4, 5, 6, 7$. Arguing as above, this implies that $W_5$ is stable for all $m \geq 2$, which matches as predicted by Hassett and Hyeon.

**Example: A genus 5 curve with a genus 2 tail.** Here we consider an example of a nodal genus 5 curve which has a genus 3 component and a genus 2 component (hence a genus 2 tail). Hassett and Hyeon predict that such a curve is stable for $m > 6$, semistable for $m = 6$, and unstable for $m < 6$. Our findings uphold this prediction.

We used the following nodal curve $C$: Let $W_3$ be the Wiman curve of genus 3, and let $P$ be the branch point at infinity. Let $W_2$ be the genus 2 Wiman curve, and let $Q$ be the branch point at infinity. Then $C = W_3 \cup_{P=Q} W_2$.

$\omega_C^2$ is very ample, and the image of $C$ under the corresponding morphism $\phi$ is a degree 16 curve in $\mathbb{P}^{11}$. We need to find equations for $\phi(C)$ and show that the action of $\mathrm{Aut}(C)$ is multiplicity-free.

We know that $\omega_C^2|_{W_3} = \omega_{W_3}^2(2P)$, and $\omega_C^2|_{W_2} = \omega_{W_2}^2(2Q)$. In principle, [24] and [8] explain how to write equations for such curves; however, we used `MAGMA` [5, 31].

For $W_3$, coordinatize $\mathbb{P}^7$ using the variables $a$–$h$, and map $W_3 \to \mathbb{P}^7$ by

$$\begin{array}{ccccccccccccccc} yx^6 & & yzx^5 & & z^5x^5 & & z^4x^6 & & z^3x^7 & & z^2x^8 & & zx^9 & & 1x^{10} \\ a & : & b & : & c & : & d & : & e & : & f & : & g & : & h \end{array}$$

Then $P$ maps to $[0:0:0:0:0:0:0:1]$. The ideal of $W_3$ in $\mathbb{P}^7$ is given by

$$\begin{aligned} &(-g^2 + fh, -fg + eh, -f^2 + eg, -f^2 + dh, -ef + dg, -ef + ch, -e^2 + df, -e^2 + cg, \\ &-de + cf, -d^2 + ce, ag - bh, af - bg, ae - bf, ad - be, ac - bd, \\ &b^2 + c^2 - fg, ab + cd - g^2, a^2 + d^2 - gh) \end{aligned}$$

and the generator of $\mathrm{Aut}(W_3)$ acts on $a$–$h$ with weights $5, 3, 10, 12, 0, 2, 4, 6$. Then the ideal of $W_3$ in $\mathbb{P}^{11}$ is obtained by adding $(i, j, k, l)$ to the ideal above, and the $\mathrm{Aut}(W_3)$-action is extended to $\mathrm{Span}\{i, j, k, l\}$ by giving these weight 6.

For $W_2$, coordinatize $\mathbb{P}^4$ using the variables $h$–$l$, and map $W_2 \to \mathbb{P}^4$ by

$$\begin{array}{ccccccccc} 1x^4 & & x^3z & & x^2z^2 & & xz^3 & & yx \\ h & : & i & : & j & : & k & : & l \end{array}$$

Then $Q$ maps to $[1 : 0 : 0 : 0 : 0]$. The ideal of $W_2$ in $\mathbb{P}^4$ is given by

$$(l^2 - hi + k^2, i^2 - hj, ij - hk, j^2 - ik),$$

and the generator of $\mathrm{Aut}(W_2)$ acts on $h$–$l$ with weights $8, 6, 4, 2, 7$. Then the ideal of $W_2$ in $\mathbb{P}^{11}$ is obtained by adding $(a, b, c, d, e, f, g)$ to this ideal, and the $\mathrm{Aut}(W_2)$-action is extended to $\mathrm{Span}\{a, b, c, d, e, f, g\}$ by giving these weight 8.

Putting this together, the ideal of $C$ is

$$\begin{aligned} I(C) \;=\; & (gl, fl, el, dl, cl, bl, al, gk, fk, ek, dk, ck, bk, \\ & ak, gj, fj, ej, dj, cj, bj, aj, gi, fi, ei, di, ci, bi, ai, \\ & g^2 - fh, fg - eh, eg - dh, dg - ch, ag - bh, f^2 - dh, ef - ch, df - cg, \\ & af - bg, e^2 - cg, de - cf, ae - bf, d^2 - ce, ad - be, ac - bd, \\ & b^2 + c^2 - eh, ab + cd - fh, a^2 + ce - gh, \\ & j^2 - ik, ij - hk, i^2 - hj, hi - k^2 - l^2) \end{aligned}$$

We check that the $\mathrm{Aut}(C)$-action is multiplicity-free. Let $\zeta_{10}$ and $\zeta_{14}$ be primitive $10^{\text{th}}$ and $14^{\text{th}}$ roots of unity, respectively. Let $K = \mathbb{Q}[\zeta_{10}, \zeta_{14}]$. Then the $\mathrm{Aut}(C)$ representation is generated by the diagonal matrices

$$D(\zeta_{14}^5, \zeta_{14}^3, \zeta_{14}^{10}, \zeta_{14}^{12}, \zeta_{14}^0, \zeta_{14}^2, \zeta_{14}^4, \zeta_{14}^6, \zeta_{14}^6, \zeta_{14}^6, \zeta_{14}^6, \zeta_{14}^6)$$

and

$$D(\zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^6, \zeta_{10}^4, \zeta_{10}^2, \zeta_{10}^7).$$

One can check in `MAGMA` [31] or `GAP` [28] that this representation is indeed multiplicity-free. Here are commands for doing this in `MAGMA` [31]:

```
K<z>:=CyclotomicField(140);
u:=z^14;
t:=z^10;
GL12K:=GeneralLinearGroup(12,K);
D1:=DiagonalMatrix([t^5,t^3,t^10,t^12,t^0,t^2,t^4,t^6,t^6,t^6,t^6,t^6]);
D2:=DiagonalMatrix([u^8,u^8,u^8,u^8,u^8,u^8,u^8,u^8,u^6,u^4,u^2,u^7]);
G:=sub<GL12K | D1,D2>;
Gmod:=GModule(G);
chi:=Character(Gmod);
X:=CharacterTable(G);
Decomposition(X,chi);
```

We ran our Monte Carlo stability program for small values of $m$. For $m < 6$, we expect that $C$ is unstable (see next paragraph), and indeed, the program timed out before finding initial ideals that spanned the barycenter. For $m = 6$, we found that $C$ is semistable. For $m = 7$, $C$ must be stable (once again this follows from the $m = 6$ calculation), but our program timed out before it found enough initial ideals to span the barycenter.

Next, we proved by hand that $C$ is unstable for $m < 6$. Let $\lambda$ be the 1-ps which acts with weights $w = 6, 6, 6, 6, 6, 6, 6, 6, 4, 2, 0, 5$. This gives average weight $\alpha = 59/12$. The Hilbert polynomial is $P(m) = 16m - 4$. The weight filtration on $\mathcal{O}(m)$ looks like

$$6m, 6m, ...6m, 6m - 1, 6m - 2, 6m - 3, ..., 6, 5, 4, 2, 0,$$

giving $w(m) = 78m^2 - 15m - 4$. Putting this all together gives

$$w(m) - mP(m)\alpha = (-2/3)(m - 1)(m - 6)$$

with the desired roots and sign for asymptotic stability with a flip at $m = 6$.

Alternatively, using the MUm function from [18] yields:

$$\begin{aligned}
\mathtt{MUm}(I, w, 2) &= -32 \\
\mathtt{MUm}(I, w, 3) &= -48 \\
\mathtt{MUm}(I, w, 4) &= -48 \\
\mathtt{MUm}(I, w, 5) &= -32 \\
\mathtt{MUm}(I, w, 6) &= 0 \\
\mathtt{MUm}(I, w, 7) &= 48
\end{aligned}$$

and then their formula (2.6), gives $\mu([C]_m, \rho) = 8(m - 1)(m - 6)$.

Note that the two calculations differ by a constant $-12$; we have not completely reconciled this yet, but under each set of authors' sign convention, the respective calculation shows that $C$ is unstable for $m < 6$.

## 8. What next?

(1) Clearly, the next step should be to prove stability for an arithmetic genus 5 curve with a ramphoid cusp for $m < 6$, and semistability for $m = 6$. We tried an irreducible example as well as a genus three curve glued to a rational component with a ramphoid cusp. The Monte-Carlo program timed out in both cases before finding enough initial ideals to span the barycenter. We hope that if we analyze the output for the genus 5 curve with a genus 2 tail more carefully, perhaps we will be able to identify some key initial ideals, and then use these to help prove stability for the ramphoidal examples.

(2) More generally, we hope that we might be able to go from random calculations to deterministic proofs. For instance, for the Wiman curves, can we identify geometrically a small number of initial ideals which span the barycenter? Then perhaps this geometrically meaningful cluster will span the barycenter for any $g$, giving us a proof in higher genus.

(3) Thanks to openness of semistability and the coarseness of the Zariski topology, the calculations presented above tell us a lot more than just stability of Wiman curves:

**Proposition 8.1** *A general smoothable curve of Hilbert polynomial $P(t) = 8t - 2$, $P(t) = 12t - 3$, or $P(t) = 16 - 4$ is stable for $m \geq 3$.*

We note that the curves need not be canonically embedded, nor smooth. (Must the linear system be complete?) We hope that techniques like those found in [7] could be used to construct moduli spaces.

(4) It would be nice to prove the Bayer–Morrison theorem for small $m$, or find a counterexample, and hence streamline Section 3.

(5) It would be nice to find more multiplicity-free curves than just bicanonical Wiman curves. Are there multiplicity-free nonhyperelliptic canonical curves? Are there any GIT stable multiplicity-free ribbons?

We know one place not to look: Bicanonical trivalent graph curves (the "vital points" of $\overline{M}_g$) are never multiplicity-free for $g < 101$. We can write equations for them, and it is easy to see that a multiplicity-free trivalent graph curve would have to be vertex and edge-transitive. Marston Conder did a search for us, and there are no such graphs, at least with $g < 101$.

(6) It would be interesting to know how the theory behaves in positive characteristic. Computing over a finite field ought to be faster than computing over $\mathbb{Q}$. Curves in positive characteristic can have automorphism groups which exceed the Hurwitz bound; perhaps this could make it easier to find multiplicity-free examples. On the other hand, representation theory in characteristic $p$ may be harder than in characteristic 0. And is knowing GIT stability in char $p$ for a small number of $p$'s enough to conclude it in char 0?

## References


[1] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, and Joe Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985. MR770932 (86h:14019) ←13

[2] Elizabeth Baldwin and David Swinarski, *A geometric invariant theory construction of moduli spaces of stable maps*, Int. Math. Res. Pap. IMRP **1** (2008), Art. ID rpn 004, 104, available at arXiv:0706.1381v3. MR2431236 ←5

[3] Dave Bayer, *The division algorithm and the Hilbert scheme*, Harvard University, 1982, http://www.math.columbia.edu/~bayer/thesis/thesis.html. ←8

[4] David Bayer and Ian Morrison, *Standard bases and geometric invariant theory. I. Initial ideals and state polytopes*, J. Symbolic Comput. **6** (1988), no. 2-3, 209–217. Computational aspects of commutative algebra. MR988413 (90e:13001) ←3, 4, 7, 8

[5] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). MR1484478 ←16

[6] Thomas Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series, vol. 280, Cambridge University Press, Cambridge, 2000. MR1796706 (2002i:14034) ←1, 10

[7] Lucia Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), no. 3, 589–660. MR1254134 (95d:14014) ←18

[8] David Eisenbud, *Transcanonical embeddings of hyperelliptic curves*, J. Pure Appl. Algebra **19** (1980), 77–83. MR593247 (82f:14027) ←13, 14, 16

[9] John Fogarty, Frances Kirwan, and David Mumford, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012) ←9

[10] Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR1785292 (2001f:52033) ←2

[11] David Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977), no. 3, 233–282. MR0498596 (58 #16687) ←1

[12] David Gieseker, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 69, Published for the Tata Institute of Fundamental Research, Bombay, 1982. MR691308 (84h:14035) ←1, 5, 16

[13] David Gieseker, *Geometric invariant theory and applications to moduli problems*, Invariant theory, 1983, pp. 45–73. Proceedings of the 1st 1982 Session of the Centro Internazionale Matematico Estivo (CIME) held at Montecatini, June 10–18, 1982. MR718124 (84i:14004) ←1

[14] Mark Haiman and Bernd Sturmfels, *Multigraded Hilbert schemes*, J. Algebraic Geom. **13** (2004), no. 4, 725–769, available at arXiv:math/0201271. MR2073194 (2005d:14006) ←2, 3

[15] Joe Harris, *Algebraic geometry: a first course*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995. Corrected reprint of the 1992 original. MR1416564 (97e:14001) ←13

[16] Brendan Hassett and Donghoon Hyeon, *Log minimal model program for the moduli space of stable curves: the first flip* (2008), available at arXiv:0806.3444v1. ←3, 11, 13, 16

[17] Brendan Hassett and Donghoon Hyeon, *Log canonical models for the moduli space of stable curves: first divisorial contraction* (2006), available at arXiv:math/0607477. To appear in Trans. Amer. Math. Soc. ←3, 11

[18] Brendan Hassett, Donghoon Hyeon, and Yongnam Lee, *Stability computation via Gröbner bases* (2007), available at http://math.rice.edu/~hassett/papers/GroebnerGIT/stabcomp.pdf. ←2, 16, 18

[19] Donghoon Hyeon and Yongnam Lee, *Log minimal model program for the moduli space of stable curves of genus three* (2007), available at arXiv:math/0703093. ←15

[20] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316. MR506989 (80c:20057) ←9, 10

[21] C. Maclachlan, *Abelian groups of automorphisms of compact Riemann surfaces*, Proc. London Math. Soc. (3) **15** (1965), 699–712. MR0179348 (31 #3596) ←11

[22] Ian Morrison, *GIT constructions of moduli spaces of stable curves and maps* (2008), available at arXiv:0810.2340v1. ←1, 5, 11

[23] Guy Rousseau, *Immeubles sphériques et théorie des invariants*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 5, A247–A250 (French, with English summary). MR0506257 (58 #22063) ←9
[24] Jan Stevens, *Deformations of singularities*, Lecture Notes in Mathematics, vol. 1811, Springer-Verlag, Berlin, 2003. MR1976016 (2004b:32049) ←13, 16
[25] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR1363949 (97b:13034) ←3
[26] A. Wiman, *Über die Doppelcurve auf den geradlinigen Flächen*, Acta Math. **19** (1895), no. 1, 63–71 (German). MR1554862 ←14

## Software Packages Referenced

[27] Matthias Franz, *Convex: a Maple package for convex geometry* (2006), available at http://www.fourier.ujf-grenoble.fr/~franz/convex/. Version 1.1.2. ←15
[28] The GAP Group, `GAP`*: Groups, Algorithms, and Programming, a system for computational discrete algebra* (2008), available at http://www.gap-system.org. Version 4.4.11. ←17
[29] Anders Jensen, `gfan`*: a software package for computing Gröbner fans and tropical varieties* (2008), available at http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html. Version 0.3. ←2, 8, 15
[30] Dan Grayson and Mike Stillman, `Macaulay 2`*: a software system for research in algebraic geometry* (2008), available at http://www.math.uiuc.edu/Macaulay2/. Version 1.1. ←2, 8, 16
[31] Computational Algebra Research Group, School of Mathematics and Statistics, University of Sydney, `MAGMA` *computational algebra system* (2008), available at http://magma.maths.usyd.edu.au/magma/. Version 2.15-1. ←16, 17
[32] Ewgenij Gawrilow and Michael Joswig, `polymake`*: a framework for analyzing convex polytopes* (2007), available at http://www.math.tu-berlin.de/polymake/. Version 2.3. ←2

Department of Mathematics, Fordham University, Bronx, NY 10458
*E-mail address*: morrison@fordham.edu

Department of Mathematics, University of Georgia, Athens, GA 30602
*E-mail address*: davids@math.uga.edu